Proposition 2.17. The class of regular languages is closed under complementation, intersection, and difference.

Proof. We are going to show closure under complementation; the other claims follow from Lemma 1.19 (a) & (c). Suppose that $L = \mathcal{L}(D)$ for some deterministic automaton $D = (\Sigma, Q, \delta, q_0, F)$. W.l.o.g., we can assume that q_0 is not in the range of δ. [Just add a new state q_0^* and let

$$\delta'(q,a) := \begin{cases} \delta(q,a) & \text{if } \delta(q,a) \neq q_0 \text{ and} \\ q_0^* & \text{otherwise.} \end{cases}$$

Then $(\Sigma, Q \cup \{q_0^*\}, \delta', q_0, F)$ accepts the same language as D and does not have q_0 in the range of its transition function.] Thus, let us assume that D has this property and define

$$\overline{D} := (\Sigma, Q, \delta, q_0, Q \backslash (F \cup \{q_0\}),$$

then we claim that $\mathcal{L}(\overline{D}) = \mathbb{W}^+ \backslash \mathcal{L}(D)$.

"$\subseteq$": Suppose $w \in \mathcal{L}(\overline{D})$, i.e., $q := \widehat{\delta}(q_0, w) \in Q \backslash (F \cup \{q_0\})$. This means that $w \neq \varepsilon$ (since $\widehat{\delta}(q_0, \varepsilon) = q_0$) and $\widehat{\delta}(q_0, w) \notin F$, so $w \notin \mathcal{L}(D)$.

"$\supseteq$": Suppose $\varepsilon \neq w$ is such that $w \notin \mathcal{L}(D)$, i.e., $\widehat{\delta}(q_0, w) \notin F$. Since $w \neq \varepsilon$, we know that $\widehat{\delta}(q_0, w) \neq q_0$ (by our assumption about the range of δ), so together, this implies that $w \in \mathcal{L}(\overline{D})$. Q.E.D.