Proposition 1.23. Let G and G' be grammars that do not share any variables and are variable-based. Let H be their concatenation grammar. Then $\mathcal{L}(H) = \mathcal{L}(G)\mathcal{L}(G')$.

Proof. For the forward direction, let $vw \in LM$, i.e., $v \in L$ and $w \in M$. By definition, we have a G-derivation $(\sigma_0, ..., \sigma_n)$ of v and a G'-derivation $(\tau_0, ..., \tau_m)$ of w. Then

$$T \xrightarrow{H}_1 SS' = \sigma_0 S' \xrightarrow{G}_1 \sigma_1 S' \xrightarrow{G}_1 \dots \xrightarrow{G}_1 \sigma_n S' = vS'$$
$$= v\tau_0 \xrightarrow{G'}_1 v\tau_1 \xrightarrow{G'}_1 \dots \xrightarrow{G'}_1 v\tau_m = vw$$

is an H-derivation of vw.

For the converse, let $T = \sigma_0 \xrightarrow{H}_1 \sigma_1 \xrightarrow{H}_1 \dots \xrightarrow{H}_1 \sigma_n = u$ be any H-derivation. Since there is only one rule involving T, we know that $\sigma_1 = SS'$. For $i \geq 1$, if $\sigma_i = x_0...x_\ell$, we define by recursion what it means that x_j *belongs to the first half in* σ_i. Our definition will be done in such a way that all variables occurring in the first half are in V and all variables occurring in the other half are in V'. If $i = 1$, we say that $S \in V$ belongs to the first half in σ_1 and $S' \in V'$ doesn't. Suppose $\sigma_i = \alpha\gamma\beta$ and $\sigma_{i+1} = \alpha\delta\beta$, i.e., σ_{i+1} is produced by an application of the rule $\gamma \to \delta$. We assumed that our grammars were variable-based, and hence γ consists only of variables. By definition of H, these must either all be from V or all from V'. In the first case, γ lies entirely in the first half; in the second case, γ lies entirely in the other half. Any symbol instance occurring in σ_{i+1} lies either in α, δ, or β. If the symbol instance is in α or β then it already occurred in σ_i, and we say that it *belongs to the first half in* σ_{i+1} if and only if it belonged to the first half in σ_i. If the symbol instance is in δ, then it *belongs to the first half in* σ_{i+1} if and only γ was entirely in the first half in σ_i (that's equivalent to $\gamma \in V^*\backslash\{\varepsilon\}$). If that's the case, we also say that the production step from i to $i+1$ belongs to the first half.

An induction shows that the symbols belonging to the first half form an initial segment of each σ_i. So, we have $u = vw$ where v is the subword of letters belonging to the first half. We now collect all production steps that belong to the first half and observe that they form a G-derivation of v from S; similarly, all production steps that do not belong to the first half form a G'-derivation of w from S'. This shows that $u = vw \in LM$. Q.E.D.