XXIV

Lectio Ultima

Twenty-forth Lecture of

Automata & Formal Languages

29 November 2022

Recap Decision problems for type 0 grammars

Word Problem $\{(w,v);\ w \in W_v\}$ unsolvable

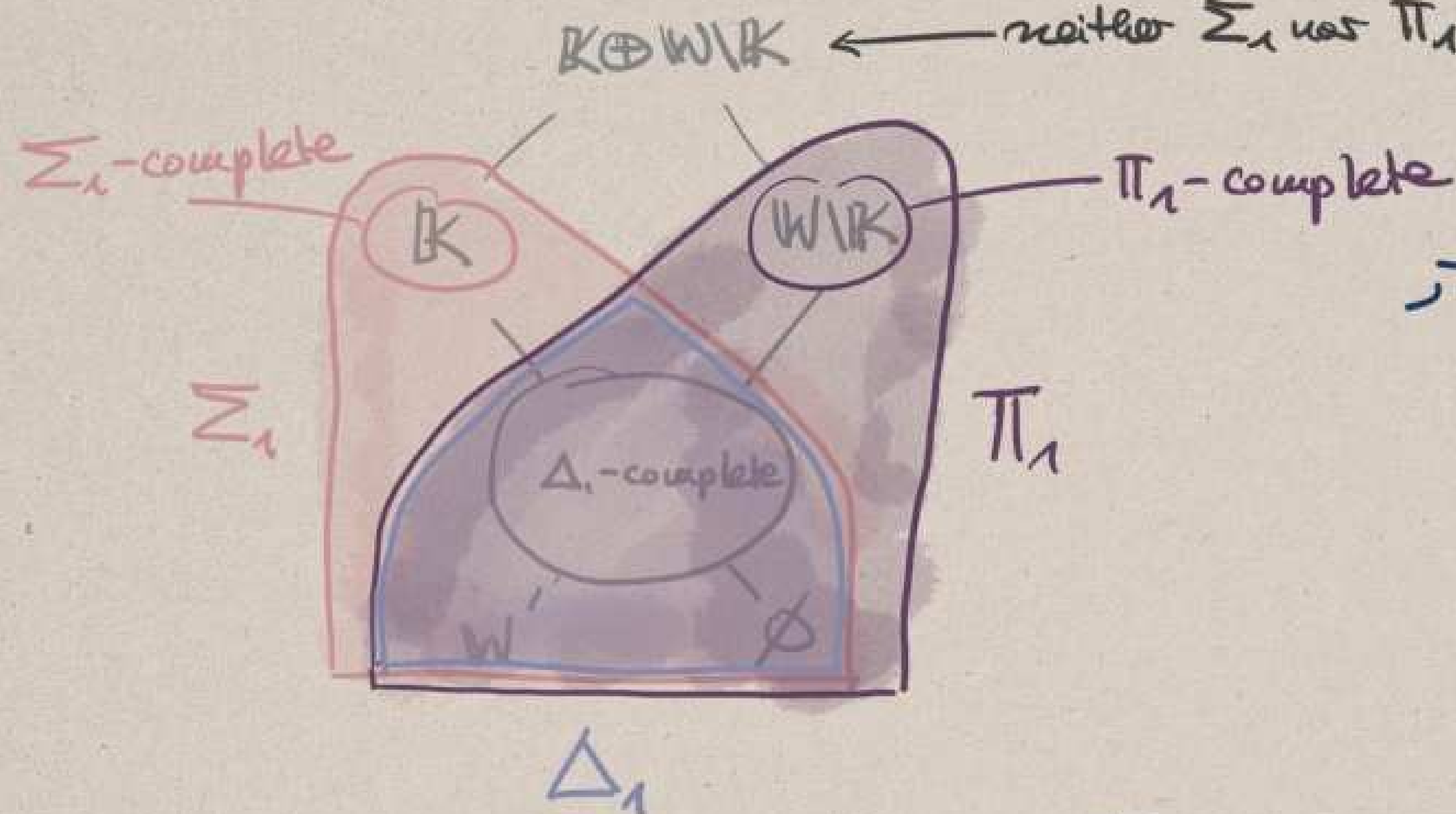
Emptiness Problem $\{v;\ W_v = \emptyset\}$?

Equivalence Problem $\{(w,v);\ W_w = W_v\}$?

Lecture $\overline{\text{XXIII}}$ Reductions, $\leq_m$ Degrees of Unsolvability.

Proof that A is not computable: $\mathbb{K} \leq_m A$.

Proof that A is not c.e.: $\mathbb{W} \setminus \mathbb{K} \leq_m A$.



§ 4.12 Index sets & Rice's Theorem

Reminder: M, M' are weakly eq. $\iff W_M = W_{M'}$

$$W_M = \mathrm{dom}(f_{M,1}), \quad W_{M'} = \mathrm{dom}(f_{M',1})$$

v, u are weakly eq. $\iff W_v = W_u$.

Write: $v \sim u$.

Def. $I \subseteq \mathbb{W}$ is called an index set if it is closed under weak eq.

$(w \in I \;\&\; w \sim v \implies v \in I)$.

(Equivalently, I is a union of $\sim$-eq. classes).

Examples Clearly, $\emptyset$ and $\mathbb{N}$ are index sets. We call them trivial.

Other index set correspond to "properties of c.e. sets":

$$\mathbb{E}\mathrm{mp} := \{v ; W_v = \emptyset\}$$

Note that this one is our EMPTINESS PROBLEM.

$$\mathbb{F}\mathrm{in} := \{v ; W_v \text{ is finite}\}$$

$$\mathbb{I}\mathrm{nf} := \{v ; W_v \text{ is infinite}\}$$

$$\mathbb{T}\mathrm{ot} := \{v ; W_v = \mathbb{N}\}$$

$[\Leftrightarrow f_{v,1} \text{ is total}]$

Theorem (Rice's Theorem) — Henry G. Rice 1920–2003

No nontrivial index set is computable.

Corollary $\mathbb{E}\mathrm{mp}$, $\mathbb{F}\mathrm{in}$, $\mathbb{I}\mathrm{nf}$, & $\mathbb{T}\mathrm{ot}$ are not computable.

Corollary The emptiness problem for type 0 grammars is unsolvable.

Proof of Rice's Thm

Preliminary discussion (remember the proof of "$\mathbb{K}$ is Σ_1-complete").

Fix some $w \in W$ and consider

$$g(u,v) := \begin{cases} f_{w,1}(v) & \text{if } f_u(u)\downarrow \quad [\Longleftrightarrow u \in \mathbb{K}] \\ \uparrow & \text{o/w} \end{cases}$$

Note: The case distinction in this construction is not computable; but if $f_u(u)\downarrow$ fails, then the machine gives the desired result, viz. $\uparrow$.

g is computable, so by the s-m-n theorem, we have total h s.t.

$$f_{h(u),1}(v) = g(u,v) = \begin{cases} f_{w,1}(v) & \text{if } u \in \mathbb{K} \\ \uparrow & \text{if } u \notin \mathbb{K}. \end{cases}$$

Analyse case 1: $u \in \mathbb{K}$.

$u \in \mathbb{K} \Longrightarrow W_{h(u)} = W_w$.

$u \notin \mathbb{K} \Longrightarrow W_{h(u)} = \emptyset$.

(*)

How do we prove that I is not computable?

Either $\mathbb{K} \leq_m I$ or

$W \setminus \mathbb{K} \leq_m I$.

Let e be s.t. $W_e = \emptyset$.

Two cases: Case 1. $e \in I$
Case 2 $e \notin I$.

<u>Case 1</u> $e \in I$. Find $w \notin I$ (nontriviality).
Take the fn g from preliminary remark with test $w \notin I$ and apply s-m-n to get total h.

<u>Claim</u> h reduces $\mathbb{N} \setminus K$ to I.

If $v \in K \overset{(*)}{\Longrightarrow} W_{h(v)} = W_w$, so $h(v) \sim w$, so $h(v) \notin I$.

If $v \notin K \overset{(*)}{\Longrightarrow} W_{h(v)} = \emptyset = W_e$, so $h(v) \sim e$, so $h(v) \in I$.

<u>Case 2</u> $e \notin I$. Find by nontriviality $w \in I$.
Take g & h as before.

<u>Claim</u> h reduces K to I.

If $v \in K \overset{(*)}{\Longrightarrow} W_{h(v)} = W_w$, so $h(v) \sim w$, so $h(v) \in I$.

If $v \notin K \overset{(*)}{\Longrightarrow} W_{h(v)} = \emptyset = W_e$, so $h(v) \sim e$, so $h(v) \notin I$.

q.e.d.

Remark The proof shows:

$$e \in \mathbb{I} \implies \mathbb{N} \setminus \mathbb{K} \leq_m \mathbb{I}$$
$$e \notin \mathbb{I} \implies \mathbb{K} \leq_m \mathbb{I}.$$

This is more information than just "not computable".

$$\left.\begin{array}{l} \mathbb{N}\setminus\mathbb{K} \leq_m \mathbb{E}\mathrm{mp} \\ \mathbb{N}\setminus\mathbb{K} \leq_m \mathbb{F}\mathrm{in} \end{array}\right\} e \in \mathbb{E}\mathrm{mp}, \mathbb{F}\mathrm{in}$$

$$\left.\begin{array}{l} \mathbb{K} \leq_m \mathbb{I}\mathrm{nf} \\ \mathbb{K} \leq_m \mathbb{T}\mathrm{ot} \end{array}\right\} e \notin \mathbb{I}\mathrm{nf}, \mathbb{T}\mathrm{ot}$$

Corollary The EQUIVALENCE PROBLEM FOR TYPE 0 GRAMMARS is not solvable

Pf. Clearly, if $W_e = \emptyset$, then

$$g : w \longmapsto (w, e)$$ can be performed by a RM

$$\mathbb{E}\mathrm{q} := \{(w, v);\ W_w = W_v\}$$

$$\chi_{\mathbb{E}\mathrm{mp}} = \chi_{\mathbb{E}\mathrm{q}} \circ g$$

Thus if $\mathbb{E}\mathrm{q}$ is computable, so is $\mathbb{E}\mathrm{mp}$.

q.e.d.

CODA ① ES#4 proves that
$\mathbb{E}mp \equiv_m \mathbb{W} \setminus \mathbb{K}$.

② The other three index sets, $\mathbb{T}ot$, $\mathbb{I}nf$, $\mathbb{F}in$ are neither Σ_1 nor Π_1.

Theorem $\mathbb{F}in$ is neither Σ_1 nor Π_1.

Proof. By (the proof of) Rice's Theorem, we know that $\mathbb{F}in$ is not Σ_1.

To show it's not Π_1, we need to show

$$\mathbb{K} \leq_m \mathbb{F}in.$$

How do we do that?

$$g(w,v) := \begin{cases} \uparrow & \text{if } t_{w,1}(w,v) = a \\ \varepsilon & \text{o/w} \end{cases}$$

Apply s-m-n to get $f_{h(w),1}(v) = g(w,v)$ & claim that h reduces $\mathbb{K}$ to $\mathbb{F}in$.

$w \in \mathbb{K} \Longrightarrow f_{h(w),1}$ is undefined from v onwards where v is the halting time of $f_w(w)$. $\Longrightarrow W_{h(w)}$ is finite $\Longrightarrow h(w) \in \mathbb{F}in$.

$w \notin \mathbb{K} \Longrightarrow f_{h(w),1}$ is the constant fn with value ε $\Longrightarrow W_{h(w)} = \mathbb{W}$ (not finite) $\Longrightarrow h(w) \notin \mathbb{F}in$.

q.e.d.