4.6 Remark on the choice of alphabet

We defined computability for partial functions $f : \mathbb{W}^k \dashrightarrow \mathbb{W}$ in terms of Σ-register machines: the instructions and behaviour of register machines are closely tied to their alphabet and register machines can only compute partial functions that use the letters that the machines are built for. Clearly, if $\Sigma \subseteq \Sigma'$ and $f : \mathbb{W}^k \dashrightarrow \mathbb{W}$ is computable by a Σ-register machine, then it is computable by a Σ'-register machine. But could it be that the notion of computability gets stronger if we add more letters to the alphabet? The answer is no as will be shown in this section.

We shall encode computations in binary notation. For this, let us assume that we have two special symbols $\mathbf{0}$ and $\mathbf{1}$ in Σ. Suppose $2 \leq n = |\Sigma|$ and k is such that $2^m \geq n$. Then we can represent the elements of Σ by binary sequences of length m by using our favourite injection i from Σ into $\{\mathbf{0}, \mathbf{1}\}^k$. The injection i induces an injection (also denoted by i) from $\mathbb{W}$ into $(\{0, 1\}^m)^* \subseteq \{\mathbf{0}, \mathbf{1}\}^* \subseteq \mathbb{W}$. We extend that induced injection further to injections $i : \mathbb{W}^n \to \mathbb{W}^n$, defined componentwise and again using the same notation.

Lemma 4.20. The injection $i : \mathbb{W} \to \mathbb{W}$ is computable and so is its inverse $i^{-1} : \mathbb{W} \dashrightarrow \mathbb{W}$ (which has domain $(\{\mathbf{0}, \mathbf{1}\}^m)^*$).

Proof. We can easily write a register machine program that removes the final letter of register k, say, a and copies $i(a)$ in reverse order into register ℓ. Repeating this until register k is empty results in the reverse of the i-image of the original content of register k to be stored in register ℓ. Now reverse the order and you obtain the i-value of the content of register k.

For the inverse, we do the same except that the program reads m many letters from the content of the register k, check that it's an element of $\{\mathbf{0}, \mathbf{1}\}^k$ (if not, we loop forever) and writes the i-preimage of that string into register ℓ. The rest of the construction is the same. Q.E.D.

Using the map i, we can represent a partial function on $\mathbb{W}$ by a partial function on $\{\mathbf{0}, \mathbf{1}\}^*$ as follows:

$$\begin{tikzcd}
\mathbb{W}^k \arrow[r, dashed, "f"] & \mathbb{W} \arrow[d, "i"] \\
(\{\mathbf{0}, \mathbf{1}\}^*)^k \arrow[u, dashed, "i^{-1}"] & \{\mathbf{0}, \mathbf{1}\}^*
\end{tikzcd}$$

Let us write $\widehat{f}$ for this partial function $i \circ f \circ i^{-1}$.

Proposition 4.21. The partial function f is computable by a Σ-register machine if and only if the partial function $\widehat{f}$ is computable by a $\{\mathbf{0}, \mathbf{1}\}$-register machine.

Proof. If M is the Σ-register machine computing f, all we need to do is to replace all instructions by sequences of instructions that do the same for the represented sequences. I.e., if the instruction is $+(\ell, a, q)$ we replace it with m many instructions that add the m bits that form $i(a)$ to register ℓ; if the instruction is $?(\ell, a, q, q')$, we replace it with a sequence of instructions that reads the final m bits from register ℓ and checks whether this sequence is $i(a)$; if the instruction is $-(\ell, q, q')$, we remove the final m bits from register ℓ instead. The instruction $?(\ell, \varepsilon, q, q')$ can remain unchanged. Q.E.D.

Corollary 4.22. Suppose $\{\mathbf{0}, \mathbf{1}\} \subseteq \Sigma \subseteq \Sigma'$ and $f : \mathbb{W}^k \dashrightarrow \mathbb{W}$ is computable by a Σ'-register machine. Then it is computable by a Σ-register machine.

Proof. We consider f as a partial function from $((\Sigma')^*)^k$ to $(\Sigma')^*$ and apply Proposition 4.21, making use of an appropriate injection $i : \Sigma' \to \{\mathbf{0}, \mathbf{1}\}^m$. This gives us a $\{\mathbf{0}, \mathbf{1}\}$-register machine that computes $\widehat{f}$. Consider $j = i{\restriction}\Sigma : \Sigma \to \{\mathbf{0}, \mathbf{1}\}^m$. The injection j and the induced injections for $\mathbb{W}$ and $\mathbb{W}^k$ as well as all of the partial inverses are computable by a Σ-register machine by Lemma 4.20. But $f = j^{-1} \circ \widehat{f} \circ j$, so f is computable by a Σ-register machine. Q.E.D.

Corollary 4.22 allows us to use the word *computable* without referring to the alphabet. It also allows us to extend the alphabet with additional letters for the convenience of proofs and show that a function $f : \mathbb{W}^k \to \mathbb{W}$ is computable by a machine using these additional letters: Corollary 4.22 tells us that these additional letters are not really needed since they can be coded away appropriately.