XVII

Seventeenth Lecture
LOGIC & SET THEORY
2 March 2026

Recap

A slightly larger arc

§ 3.5 Wellorders
The natural domains for induction and recursion

FUNDAMENTAL THEOREM ON WELLORDERS
FTOWO : Wellorders are wellorders

§ 3.6 Ordinals
The canonical representatives of wellorders
Transfinite induction & recursion

§ 3.7 The Axiom of Choice

REMINDER

f is a choice function for x if $\mathrm{dom}(f) = x$ and for all $y \in x$
$y \neq \emptyset \Rightarrow f(y) \in y$

Lemma 1 AC is equivalent to "all power sets have choice functions".

Pf. Observe that if $x \subseteq x'$ and f is c.f. for x', then $f \restriction x$ is c.f. for x.
But for every set x,
$x \subseteq \mathcal{P}(\bigcup x)$.
[By definition of $\mathcal{P}$ and $\bigcup$.]


Ernst Zermelo

Ernst Zermelo in the 1920s

Born	27 July 1871 Berlin, German Empire
Died	21 May 1953 (aged 81) Freiburg im Breisgau, West Germany

Zermelo's Wellordering Principle
ZWOP
Every set is wellorderable
↑
there is $<$ s.t. $(X, <)$ is a wellorder

It will turn out that AC is equivalent to ZWOP
→ Lecture XVII.

Lemma 1
AC eq. to "all power sets have choice fns"

Def. WELLORDERABLE
"wobble"
"wooble"

ZWOP

f is a choice fn for X if $dom(f) = X$ and $f.a.\ A \subseteq X, A \neq \emptyset$ $f(A) \in A$

Lemma 1 AC is equivalent to "all powersets have c.f."

Lemma 2

If X is w'obbe, then $\wp(X)$ has a choice function.

Corollary

ZWOP $\Longrightarrow$ AC

[Immediate from L 1 & 2.]

Proof of L2 Let $<$ be s.t. $(X, <)$ is a wellorder.

Define $f: \wp(X) \longrightarrow X$ by

Fix $x \in X$.

$$f(A) := \begin{cases} \text{the } <\text{-least element of } A & \text{if } A \neq \emptyset \\ x & \text{if } A = \emptyset. \end{cases}$$

This is a choice function. q.ed.

Lemma 3 Suppose $X \neq \emptyset$, Y sets.

Write (a) for "there is an inj. from X to Y" and (b) for "there is a surj. from Y to X".

Then (a) $\Rightarrow$ (b) and if Y is woble, then (b) $\Rightarrow$ (a).

Proof. (a) $\Rightarrow$ (b) $\ast$ N&S.

(b) $\Rightarrow$ (a). If $g: Y \to X$ is surjection, then for all $x \in X$ $G_x = \{y \in Y ; g(y) = x\} \neq \emptyset$.

Let f be the c.f. for $P(Y)$

Then $x \longmapsto f(G_x)$ is an injection from X to Y qed

Lemma 2 If X is wellorderable, then $P(X)$ has a choice fn.

Lemma 4 TFAE

(i) X is wooble
(ii) X is in bijection with ordinal
(iii) there is an injection of X into an ordinal
(iv) there is a surjection of an ordinal onto X

pf. (i) $\Rightarrow$ (ii) Representation Thm.
(ii) $\Rightarrow$ (iii) trivial
(iii) $\Leftrightarrow$ (iv) Lemma 3.

So show (iii) $\Rightarrow$ (i): $f: X \to \alpha$; consider $R = \mathrm{ran}(f)$.

Then $(R, \in)$ is a wellorder.

Define $x < y$ iff $f(x) \in f(y)$. By def. f is an iso between $(X, <)$ and $(R, \in)$. So $(X, <)$ is a wellorder. qed.

Theorem ZERMELO'S THEOREM

$AC \Longrightarrow ZWOP.$

Proof Proof is similar to that of FToWO.

Pick X arbitrary. Let $\alpha := \aleph(X)$.
Fix f choice fn for $\mathcal{P}(X)$.
STOP $\notin X$.
We define $g : \alpha \longrightarrow X \cup \{\text{STOP}\}$ by recursion

$$g(\gamma) := \begin{cases} f(X \setminus \operatorname{ran}(g\restriction\gamma)) & \text{if } X \not\subseteq \operatorname{ran}(g\restriction\gamma) \\ \text{STOP} & \text{if } X \subseteq \operatorname{ran}(g\restriction\gamma) \end{cases}$$

If STOP $\notin \operatorname{ran}(g)$, then g is by definition an inj. from α to X which doesn't exist. CONTRADICTION.

So STOP $\in \operatorname{ran}(g)$. Let γ be least s.t. $g(\gamma) = \text{STOP}$

Then $g\restriction\gamma$ is an injection

$$g\restriction\gamma : \gamma \longrightarrow X$$

but since $g(\gamma) = \text{STOP}$, have $X \subseteq \operatorname{ran}(g\restriction\gamma)$

so $g\restriction\gamma$ is also a surjection

Together by L4, X is wo'able.

q.e.d.

Ernst Zermelo



Ernst Zermelo in the 1900s

Born	27 July 1871 Berlin, German Empire
Died	21 May 1953 (aged 81) Freiburg im Breisgau, West Germany

ZORN'S LEMMA

$\mathbb{P} = (P, \leq)$ partial order

$A \subseteq P$

p is a **boound** of A if $\forall a.\ a \in A \Rightarrow a \leq p$

in this case we say A is **boounded**

A is a **chain** if $(A, \leq)$ is linear

$\mathbb{P}$ is called **chain-complete** if every chain is bounded

Theorem (Zorn's Lemma)

AC $\longrightarrow$ every nonempty chain-complete partial order has a maximal element

$\uparrow$

ZL

Versions of ZL were proved before Zorn by KURATOWSKI, HAUSDORFF, & BOCHNER

Max August Zorn



Zorn in 1930

Born	June 6, 1906 Krefeld, Rhenish Prussia, Germany
Died	March 9, 1993 (aged 86) Bloomington, Indiana, U.S.
Alma mater	University of Hamburg
Known for	Zorn ring Zorn's lemma Zorn's vector-matrix algebras Artin–Zorn theorem

Proof of Zorn's Lemma.

Again, similar to FToWO.

Let $f : \mathcal{P}(P) \longrightarrow P$ be a choice fn,
$\alpha := \aleph(P)$, fix $p \in P$ $[P \neq \emptyset]$

If $q \in P$, write $S_q := \{q' \mid q < q'\}$
q is maximal iff $S_q = \emptyset$.

If $A \subseteq P$ write $B_A := \{q \mid q$ is a bound of $A\}$
By ass., if A chain, then $B_A \neq \emptyset$.

Define by transfinite recursion $g : \alpha \longrightarrow P \cup \{\text{STOP}\}$

$$g(0) := p$$

$$g(\gamma+1) := \begin{cases} f(S_{g(\gamma)}) & \text{if } S_{g(\gamma)} \neq \emptyset \\ \text{STOP} & \text{otherwise} \end{cases}$$

$$g(\lambda) := \begin{cases} f(B_{\mathrm{ran}(g \restriction \lambda)}) & \text{if } \lambda \text{ is limit and } \mathrm{ran}(g \restriction \lambda) \subseteq P \\ \text{STOP} & \text{otherwise} \end{cases}$$

If $\text{STOP} \notin \mathrm{ran}(g \restriction \beta)$, then $g \restriction \beta$ is order preserving & injective.

So in particular $\text{STOP} \in \mathrm{ran}(g)$.
If δ is the least ordinal s.t. $g(\delta) = \text{STOP}$,
then by definition δ is successor, so $\delta = \gamma + 1$,
thus $S_{g(\gamma)} = \emptyset$, so $g(\gamma)$ is maximal.

q.e.d.

Theorem ZL $\Rightarrow$ AC.

Proof. By L1, we need to find a c.f. for every power set. Fix X and $\wp(X)$ and consider

$$P := \{f \mid f \text{ is a c.f. on some } Y \subseteq \wp(X)\}$$

This is nonempty

Define $f \leq g$ iff $f \subseteq g$, i.e.,

Δ f is c.f. on Y $\quad Y \subseteq Z$
is c.f. on Z
and $g \restriction Y = f$.

This is chain-complete: if C is chain, then $\bigcup C$ is a c.f. s.t. $\forall g \in C,\ g \subseteq \bigcup C$

So $\bigcup C$ is a bound for C.

By ZL, $(P, \leq)$ has a maximal element. Call it f.

Claim $\mathrm{dom}(f) = \wp(X)$. [Suppose $A \notin \mathrm{dom}(f)$. if $A = \emptyset$, let $z := \emptyset$; if $A \neq \emptyset$, let $z \in A$. Define $f \cup \{(A, z)\}$. This is a c.f. extending f in contradiction to maximality qed

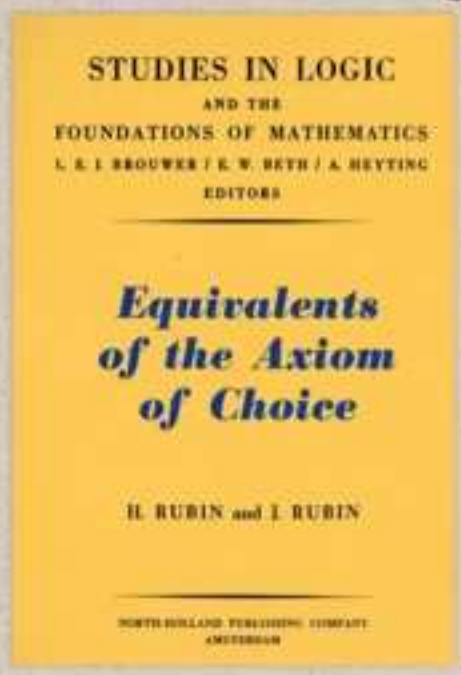


Emil Artin

Born	March 3, 1898 Vienna, Austria-Hungary
Died	December 20, 1962 (aged 64) Hamburg, West Germany
Alma mater	University of Vienna University of Leipzig
Known for	Abstract algebra List of things named after Emil Artin

According to Zorn's recollections, the proof of ZL $\Rightarrow$ AC is due to Emil Artin.

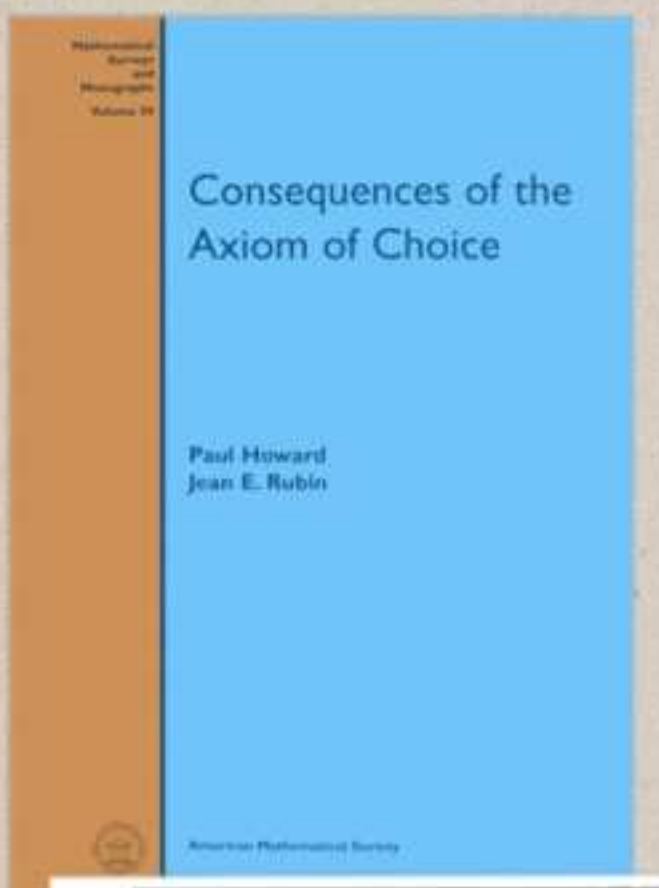
The Use of the Axiom of Choice





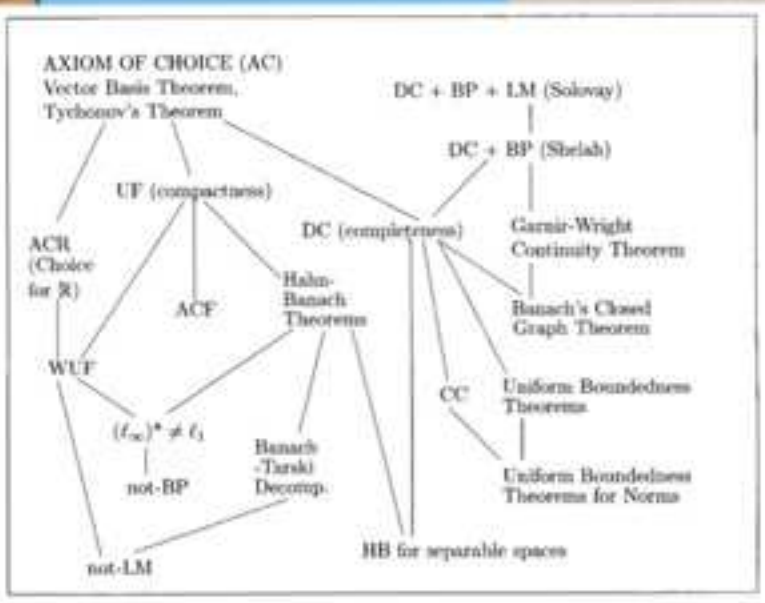
Equivalent to the Axiom of Choice:

1. "Every vector space has a basis"
2. "Every ideal in a ring can be extended to a maximal ideal."
3. TYCHONOFF: Products of compact spaces are compact.



Not ZF-theorems, but not equivalent to AC:

1. There is a non-Lebesgue measurable set.
2. Every field has an algebraic closure.
3. Hahn-Banach theorem.
4. Stone representation theorem: Every Boolean algebra is isomorphic to a subalgebra of a power set algebra.



partly on a chart of Pincus [1974]. All assertions in the chart are understood to be in conjunction with ZF. Implications in the chart are downward, and we shall prove most of these implications. For instance, in this chapter we include proofs of AC ⇒ DC ⇒ CC

131

ERIC SCHECHTER
Handbook of Analysis & its Foundations