XIX NINETEENTH LECTURE

Logic & Set Theory

6 March 2026

Cardinals

Initial ordinal / Cardinal

$|X| :=$ unique initial ordinal α s.t. there is a bij. between X & α

[needs AC / ZWOP]

Friedrich Moritz Hartogs



Friedrich Hartogs

Born	20 May 1874 Brussels
Died	18 August 1943 (aged 69) Munich

$\kappa \longmapsto \aleph(\kappa) = \kappa^+$

$\kappa \longmapsto \beth(\kappa)$

Both produce strictly larger cardinals

$\kappa^+ \leq |\beth(\kappa)|$

Georg Cantor



Cantor, c. 1910

Born	Georg Ferdinand Ludwig Philipp Cantor 3 March 1845 Saint Petersburg, Russian Empire
Died	6 January 1918 (aged 72) Halle, Province of Saxony, German Empire



Alephs

$\aleph_0 := \omega$

$\aleph_{\alpha+1} := \aleph(\aleph_\alpha)$

$\aleph_\lambda := \bigcup \{\aleph_\alpha ; \alpha < \lambda\}$ λ limit

Cardinal addition and multiplication $\neq$ Ordinal addition & multiplication

disjoint union

product

$\aleph_0 + \aleph_0 = \aleph_0 = \omega < \omega + \omega$

$\aleph_0 \cdot \aleph_0 = \aleph_0 = \omega < \omega \cdot \omega$

Thm (Hessenberg's Thm; ZF)

For all infinite cardinals κ,

$$|\kappa \times \kappa| = |\kappa|.$$

[Note: AC not needed if stated for CARDINALS, & stated for arbitrary sets, then it implies AC.]

Proof: Next time.

Lecture XIX.

Corollary

If $2 \leq \kappa \leq \lambda$ and λ infinite

then

$$\lambda \leq \kappa + \lambda \leq \kappa \cdot \lambda \leq \lambda \cdot \lambda = \lambda.$$

So $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$.

⇝ Cardinal addition & multiplication for infinite cardinals trivialize.



Gerhard Hessenberg

Hessenberg in May 1927

Born	8 August 1874 Frankfurt
Died	18 November 1925 (aged 51) Bonn
Alma mater	University of Berlin
Known for	Hessenberg sum and product
Scientific career	
Fields	Mathematics
Institutions	University of Breslau
Thesis	*Über die invarianten linearer und quadratischer binärer Differentialformen und ihre Anwendung auf die Deformation der Flächen* (1899)
Doctoral advisor	Hermann Schwarz
Other advisors	Leopold Fuchs

Proof of Hessenberg's Thm

Let ξ be an ordinal and define an order on $\xi \times \xi$.

$$(\alpha, \beta) < (\gamma, \delta)$$

iff

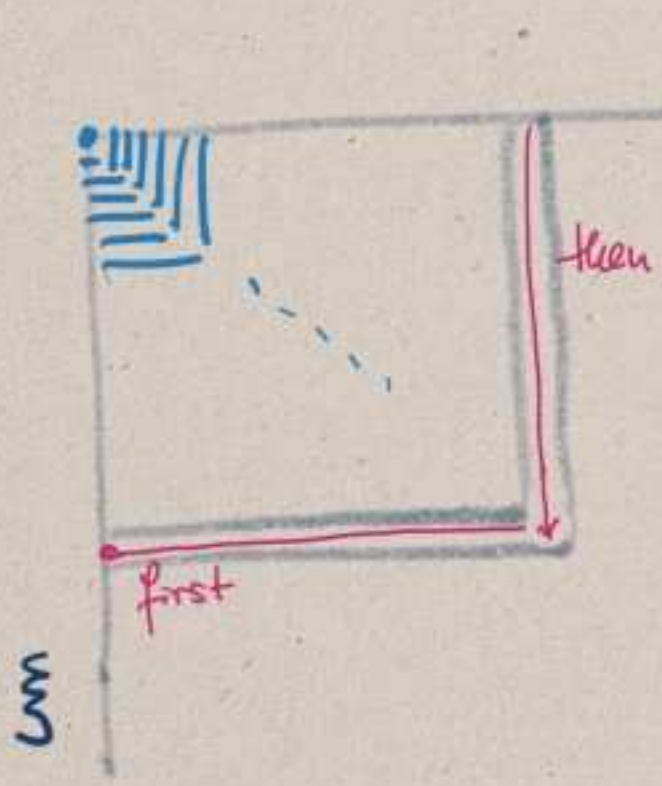
$$\max(\alpha, \beta) < \max(\gamma, \delta)$$

OR

$$\left[\max(\alpha, \beta) = \max(\gamma, \delta) \text{ AND } \alpha < \gamma\right]$$

OR

$$\left[\max(\alpha, \beta) = \max(\gamma, \delta) \text{ AND } \beta < \delta\right]$$



Facts about $<$:

① $<$ is a wellorder on $\xi \times \xi$

Let γ_ξ be the unique ordinal s.t. $\exists g_\xi : (\xi \times \xi, <) \cong (\gamma_\xi, \in)$.

↖ isomorphism

② Then $\gamma_\xi \geq \xi$

③ Also $\gamma_\omega = \omega$.

④ If $\xi \leq \eta$, then $g_\eta \restriction \xi \times \xi = g_\xi$.

[Drop subscript of g.]

We'll show: If $\kappa \subseteq$ infinite initial ordinal, then $\gamma_\kappa = \kappa$.

[This proves Hessenberg.]

Proof of claim: $\gamma_\kappa = \kappa$.

Let κ be the least initial ordinal s.t.

$\gamma_\kappa \neq \kappa$, so $\gamma_\kappa > \kappa$.

By def., this means: there are $\alpha, \beta < \kappa$ s.t. $g(\alpha,\beta) = \kappa$. Let $\delta := \max(\alpha,\beta)+1$.

Since κ is initial ordinal,

$$|\delta| < \kappa.$$

By IH, we have $|\delta| = |\delta \times \delta|$.

By choice of δ, $g(\alpha,\beta) \in \gamma_\delta$.

$$\kappa$$

Thus $|\gamma_\delta| \geq \kappa$.

Put together:

$$\kappa \leq |\gamma_\delta| \overset{\text{def}}{=} |\delta \times \delta| = |\delta| < \kappa$$

Contradiction!

q.e.d.

Cardinal exponentiation

$$\kappa^\lambda := |\{f \mid f \in \mathcal{P}(\lambda \times \kappa);\ f : \lambda \to \kappa\}|$$

Remarks. (1) Need the Axiom of Choice to define this.

[Contrast w/ + and · !]

(2) Laws of exponentiation:

$$(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$$

$$\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$$

$$\kappa^{\lambda \cdot \mu} = (\kappa^\lambda)^\mu$$

$\kappa \leq \lambda \Rightarrow \kappa^\mu \leq \lambda^\mu$ and $\mu^\kappa \leq \mu^\lambda$.

[ES#4: Check that the obvious fns are bijections & injections.]

(3) Very diff. from ordinal exponentiation

$\aleph_0^{\aleph_0}$ is uncountable, but

$$\omega^\omega = \bigcup_{n \in \omega} \omega^n \text{ is countable.}$$

(4) Special case: $\kappa = 2$.

Then 2^λ is in bijection with $\mathcal{P}(\lambda)$.

(5) $2^{\aleph_0} = |\mathbb{R}|$.

[Represent $\mathbb{R}$ as binary seq.'s, get $2^{\aleph_0} \leq |\mathbb{R}|$ and $|\mathbb{R}| \leq 2^{\aleph_0}$ and use CSB.]

$\lambda \longmapsto 2^\lambda$ is called the CONTINUUM FUNCTION.

not a function!

Lemma κ cardinal, λ infinite cardinal
if $2 \leq \kappa \leq \lambda^+$, then $2^\lambda = \kappa^\lambda$.

Proof. $2^\lambda > \lambda$ by Cantor, so $2^\lambda \geq \lambda^+$.

$$2^\lambda \leq \kappa^\lambda \leq (\lambda^+)^\lambda \leq (2^\lambda)^\lambda = 2^{\lambda \cdot \lambda} = 2^\lambda$$

[Hessenberg]

q.e.d.



BETH
2nd letter of the Hebrew alphabet

Beth hierarchy

$\beth_0 := \omega$

$\beth_{\alpha+1} := 2^{\beth_\alpha}$

$\beth_\lambda := \bigcup \{\beth_\alpha \mid \alpha < \lambda\}$

By induction (since $\kappa^+ \leq 2^\kappa$), $\aleph_\alpha \leq \beth_\alpha$.

What is the relationship between the two hierarchies.

More concretely: $\aleph_1$ and $\beth_1 = 2^{\aleph_0}$.

⟶ Cantor's Continuum Problem.

Is $\aleph_1 = \beth_1$?

The Continuum Problem

David Hilbert



Hilbert in 1912

Born	23 January 1862 Königsberg or Wehlau, Kingdom of Prussia
Died	14 February 1943 (aged 81) Göttingen, Nazi Germany
Education	University of Königsberg (PhD)

Mathematische Probleme.

Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900.

Von

D. Hilbert.

Wer von uns würde nicht gern den Schleier lüften, unter dem die Zukunft verborgen liegt, um einen Blick zu werfen auf die bevorstehenden Fortschritte unserer Wissenschaft und in die Geheimnisse ihrer Entwickelung während der künftigen Jahrhunderte! Welche besonderen Ziele werden es sein, denen die führenden mathematischen Geister der kommenden Geschlechter nachstreben? welche neuen Methoden und neuen Thatsachen werden die neuen Jahrhunderte entdecken — auf dem weiten und reichen Felde mathematischen Denkens?



Hilbert was asked to provide a problem list for the 20th century.

23 Hilbert problems

#1: Continuum Problem

Hilbert chose to give the continuum problem as the very first problem on his list!

A preview of Part III

"Forcing & the Continuum Hypothesis"
[Part III course]

Kurt Gödel



Gödel c. 1926

Born	Kurt Friedrich Gödel April 28, 1906 Brünn, Austria-Hungary (now Brno, Czech Republic)
Died	January 14, 1978 (aged 71) Princeton, New Jersey, U.S.
Citizenship	Austria Czechoslovakia Germany United States
Alma mater	University of Vienna (PhD, 1930)

Thm (1938)

If there is a model $M \models ZF$,
then there is M^* with $M^* \subseteq M$
(inner model)
s.t. $M^* \models ZFC + CH$
$+ \forall \alpha \; \aleph_\alpha = \beth_\alpha$
generalised CH
GCH

Paul J. Cohen



Paul J. Cohen

Born	April 2, 1934 Long Branch, New Jersey, U.S.
Died	March 23, 2007 (aged 72) Stanford, California, U.S.
Alma mater	University of Chicago (MS, PhD)

Thm (1962)

If there is a model $M \models ZFC$
then there is $M^* \models ZFC + \neg CH$.

Together: CH is independent of the axioms of ZFC

If ZFC is consistent, then
$ZFC \not\vdash CH$
and
$ZFC \not\vdash \neg CH$.

A preview of results BEYOND Part III

Thm (Easton) and order pres.

If F is any strictly increasing ordinal operation, i.e., $F(\alpha) > \alpha$, then if there is $M \models ZFC$, then there is M^* s.t. in M^* for all α

$$2^{\aleph_\alpha} = \aleph_{F(\alpha)}$$

William Bigelow Easton

1 language

Article Talk Read Edit source View history

From Wikipedia, the free encyclopedia



William Bigelow Easton was an American mathematician who proved Easton's theorem about the possible values of the continuum function. His advisor at Princeton was the famed mathematician and computer scientist Alonzo Church.[1]

Publications [edit source]

- Easton, W. (1970), "Powers of regular cardinals", *Ann. Math. Logic*, 1 (2): 139–178, doi:10.1016/0003-4843(70)90012-4

References [edit source]

1. "William Bigelow Easton". Mathematics Genealogy Project. Retrieved 3 November 2022.

For limit cardinals, things are more complicated.

Thm (Silver)

If $\forall \alpha < \omega_1$ $2^{\aleph_\alpha} = \aleph_{\alpha+1}$) then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.

Jack Silver



Jack Silver in 1986
(photo by George Bergman)

Born	Jack Howard Silver April 23, 1942 Missoula, Montana, U.S.
Died	December 22, 2016 (aged 74)
Alma mater	University of California, Berkeley
Known for	Silver forcing

Saharon Shelah
שהרן שלח



Shelah in 2005

Born	July 3, 1945 (age 80) Jerusalem, British Mandate for Palestine (now Israel)
Alma mater	Tel Aviv University (B.Sc) Hebrew University (M.Sc., Ph.D.)
Known for	Proper Forcing, PCF theory, Sauer–Shelah lemma, Shelah cardinal
Awards	Erdős Prize (1977) Rothschild Prize (1982) Karp Prize (1983) George Pólya Prize (1992) Gödel Lecture (1996) Bolyai Prize (2000) Wolf Prize (2001) Israel Prize (1998) EMET Prize (2011) Leroy P. Steele Prize (2013) Rolf Schock Prize (2018)

NEXT TIME (Lecture XX): the final section of Chapter 3 on the axiom of foundation