XIII Thirteenth lecture of LOGIC & SET THEORY

20 February 2026

§3.4 Induction & Recursion (Replacement)

We are working in ZF = Z + Repl.

Induction ORDER INDUCTION vs COMPLETE INDUCTION

Order induction is equivalent to wellfoundedness.

There are structures such as $S(\mathbb{N}) = \mathbb{N} \cup \{\mathbb{N}\}$ that satisfy order induction but not complete induction

Recursion

Lecture XII

Theorem (Recursion Theorem)

Suppose X is a wellfounded strict partial order.

Let φ be a fmla with free variables x, y, z with the following properties:

functional: for all a, b if $\varphi(a,b,c)$ and $\varphi(a,b,c')$, then $c = c'$.

total: for all a, b there is a c s.t. $\varphi(a,b,c)$

Then there is a unique F s.t. $\mathrm{dom}(F) = X$ and for all $x \in X$

RECURSION CONDITION (*) $F(x) = z$ iff $\varphi(x, F\restriction \mathrm{pred}(x), z)$

We started the proof of the recursion theorem.

Theorem (Recursion Theorem)
Suppose X is a well-founded strict partial order.
Let φ be a formula with free variables x, y, z with the following properties:

functional: for all a, b if $\varphi(a,b,c)$ and $\varphi(a,b,c')$, then $c = c'$

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RECURSION CONDITION $(*)$ $F(x) = z$ iff $\varphi(x, F\restriction \mathrm{pred}(x), z)$

Proof. Say $Y \subseteq X$ is downwards closed (dwc) if
$\forall z, y \,(z < y \wedge y \in Y \Rightarrow z \in Y)$
Call f an attempt if $\mathrm{dom}(f)$ is a dwc subset of X and it satisfies $(*)$ on its domain

Note $\emptyset$ is an attempt and if f if there is an attempt with $\mathrm{dom}(f) = X$, we have proved existence.

Claim 1 If f, g are attempts and $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ then $f(x) = g(x)$.

[Follows from order induction and $(*)$.]

Claim 1 implies the uniqueness claim of the theorem.

Let us now complete the proof of the recursion theorem:

CLAIM 2 If Y is dwc and for each $y \in Y$ there is an attempt g s.t. $y \in \mathrm{dom}(g)$ then there is an attempt h with $\mathrm{dom}(h) = Y$.

[Let $\psi(x,z)$ be the formula

"there is an attempt f with $x \in \mathrm{dom}(f)$ and $f(x) = z$"

CONVENTION for "the obvious $\in$-formula which expresses this"

By Claim 1, ψ is functional.

Use Replacement to find r s.t.

for all $y \in Y$ if $\psi(y,z)$, then $z \in r$

That's where Replacement is used!

Define by Separation:

$h := \{(y,z) \in Y \times r ;\ \psi(y,z)\}$

That's an attempt with $\mathrm{dom}(h) = Y$.]

CLAIM 3 For all $x \in X$ there is an attempt f with $x \in \mathrm{dom}(f)$.

NOTE: Claims 2&3 give existence.

[Induction. Suppose not, then there is some $m \in X$ $<$-minimal s.t. m is not in the domain of att: m is not in the dom. of att. but every $x \in \mathrm{pred}(m)$ is!

Apply Claim 2 to $\mathrm{pred}(m)$, so find g attempt with $\mathrm{dom}(g) = \mathrm{pred}(m)$. Find z s.t. $\varphi(m,g,z)$ [exists by Totality]. Define $h := g \cup \{(m,z)\}$. Contradiction!]

q.e.d.

Remark on the use of REPLACEMENT

We only used Replacement in the proof of Claim 2 in order to get the range of the attempt (in general of F itself).

If that range is known in advance, e.g. $\varphi(a,b,c) \Longrightarrow c \in r$ for some fixed r, then this can be proved without Replacement.

In particular, the GRASSMANN recursion equations for $+$ and $\cdot$ on $\mathbb{N}$ give a recursion theorem (and thus a uniquely existing function) in FST + Inf.

Example 1

There is an inductive set $\neq \mathbb{N}$.

Take any $x \notin \mathbb{N}$ and define

$$F(0) := x$$
$$F(s(k)) := s(F(k))$$

By RT, this defines a function F with $\mathrm{dom}(F) = \mathbb{N}$. Let's define Let $R_x := \mathrm{ran}(F)$ and

define $a := \mathbb{N} \cup R_x$

Clearly $a \neq \mathbb{N}$, since $x \in a$.

Clearly a is inductive.

I get LOTS OF inductive sets.

Depending on what x is, the existence of a may not be provable in $\mathbb{Z}$.

In particular, if $x = \mathbb{N}$, the resulting inductive set may not exist in models of $\mathbb{N}$.

Example 2

For all x, there is a smallest transitive $t \supseteq x$.

[i.e., if $t' \supseteq x$ trs, then $t' \supseteq t$.]

We call it the transitive closure of x:

$tcl(x)$.

NOTE
Corrected from lectures: the $\bigcup$ symbol was missing!

Existence proof:-

$$F(0) := x$$
$$F(s(k)) := \bigcup F(k)$$

By RT, F exists and has a range

$$tcl(x) := \bigcup \mathrm{ran}(F).$$

Transitive

$z \in y \in tcl(x)$
$\rightsquigarrow$ there is k s.t.
$z \in y \in F(k)$
$\rightsquigarrow z \in \bigcup F(k) = F(s(k))$
so $z \in tcl(x)$.

Smallest

Suppose $t' \supseteq x$ transitive.
Prove by induction on k that
$F(k) \subseteq t'$.
$\rightsquigarrow tcl(x) \subseteq t'$.

In particular: Every set is contained in a transitive set.

§3.5 Wellorders

We work in a model of ZF.

A wellfdd strict linear order is called a

WELLORDER

In a wellorder, we call dwc sets initial segments.
If X is the wo and $I \subseteq X$ is an initial segment s.t. $I \neq X$, we say it's proper

PRIME Ex. $(\mathbb{N}, <)$

PRIME NON-Ex. $(\mathbb{Z}, <)$, $(\mathbb{Q}, <)$, $(\mathbb{R}, <)$

Prop. If X wellorder, $A \subseteq X$ then TFAE
(i) A proper in. seg.
(ii) there is x s.t. $A = \text{pred}(x)$.

Pf. $\Leftarrow$ obv. $\Rightarrow$: if $X \setminus A \neq \emptyset$ and m is $<$-min. in $X \setminus A$, then $A = \text{pred}(m)$.
q.e.d.

Examples (1) If X is a wellorder and $A \subseteq X$, define $<_A := < \cap (A \times A)$. Then $(A, <_A)$ is a wellorder.

(2) Therefore, if X wo and $x \in X$, then $\text{pred}(x)$ is a wellorder.

Examples (continued)

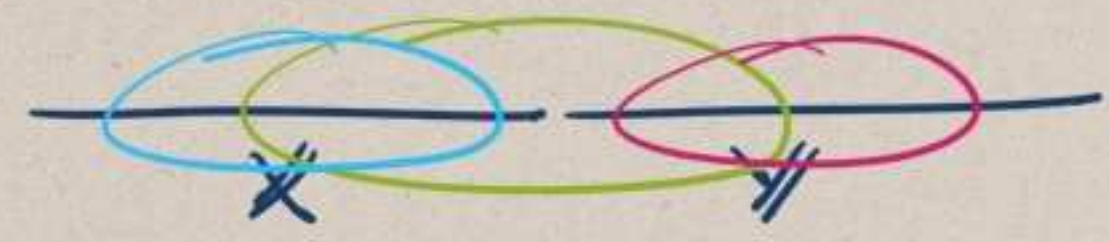
If $\mathbb{X}$, $\mathbb{Y}$ are wellorders, let's define their sum $\mathbb{X} \oplus \mathbb{Y}$.

Assume $X \cap Y = \emptyset$ and define $<^+$ on $X \cup Y$.

On $X \cup Y$, define the order $<^+$ by

$$a <^+ b \text{ if and only if } \begin{cases} a \in X, b \in X, \text{ and } a < b, \text{or} \\ a \in Y, b \in Y, \text{ and } a < b, \text{ or} \\ a \in X \text{ and } b \in Y, \end{cases}$$

Picture:



We observe: this is a wellorder.

Two cases: $A \cap X \neq \emptyset$, then the $<$-min. elt. of $A \cap X$ is $<^+$-minimal

$A \cap X = \emptyset$, so $A \subseteq Y$, then the $<$-min. elt. of $A \cap Y = A$ is $<^+$-min.

If we write $\mathbb{1} = (1, <)$, then

$S(\mathbb{N})$ is isomorphic to $\mathbb{N} \oplus \mathbb{1}$.