XII TWELFTH LECTURE

(the midpoint of the course)

LOGIC & SET THEORY

18 February 2026

Recap

FST reconstructed "abstract maths".

We're reconstructing in FST+Inf "concrete maths" (for convenience $\mathbb{Z}$).

$\mathbb{N}$: smallest inductive set

its elements: natural numbers

$0 := \emptyset$

$1 := s(0)$

$2 := s(1) = s(s(0))$

$3 := s(2) = s(s(s(0)))$

← Q: Does this mean that all nat. numbers are of the form $s(\dots s(0))$.

$\mathbb{N}$ and $n \in \mathbb{N}$ are transitive sets

$\mathbb{N}$ satisfies complete induction:

$A \subseteq \mathbb{N}, 0 \in A, x \in A \Rightarrow s(x) \in A$

$\rightsquigarrow A = \mathbb{N}$

$\in$ is a wellfounded strict linear order on $\mathbb{N}$

Recursion

$\rightsquigarrow +, \cdot$

Since induction implies recursion (for more details, see Lecture XII)

define by GRASSMANN Recursion Equations

$n+0 := n$ $\quad$ $n+s(m) := s(n+m)$

$n \cdot 0 := 0$ $\quad$ $n \cdot s(m) := n \cdot m + n$

Lemma 3.11. Let $\mathbf{X} = (X, +, \cdot, 0, 1, <)$ be a structure.

(i) Define $\sim$ on X^2 by $(x, y) \sim (x', y')$ if and only if $x + y' = y + x'$. If $\mathbf{X}$ is an ordered commutative semiring, then $\mathbf{X}^{\oplus} := (X^2/\sim, +, \cdot, [(0,0)], [(1,0)], <)$ is an ordered commutative ring where

$$[(x,y)] + [(x',y')] := [(x+x', y+y')],$$
$$[(x,y)] \cdot [(x',y')] := [(xx'+yy', xy'+x'y)], \text{ and}$$
$$[(x,y)] < [(x',y')] \text{ if and only if } x + y' < y + x'.$$

(ii) Define $\sim$ on X^2 by $(x, y) \sim (x', y')$ if and only if $x \cdot y' = y \cdot x'$ and $X^* := \{x \in X : x > 0\}$. If $\mathbf{X}$ is an ordered commutative ring, then $\mathbf{X}^{\otimes} := ((X \times X^*)/\sim, +, \cdot, [(0,1)], [(1,1)], <)$ is an ordered field where

$$[(x,y)] + [(x',y')] := [(xy' + x'y, yy')],$$
$$[(x,y)] \cdot [(x',y')] := [(xx', yy')], \text{ and}$$
$$[(x,y)] < [(x',y')] \text{ if and only if } xy' < x'y.$$

$\rightsquigarrow \mathbb{Z}$ ring of integers

$\rightsquigarrow \mathbb{Q}$ field of rationals

UNDERLYING SETS: $\mathbb{Z}$ & $\mathbb{Q}$

SIDE REMARK

Q: Can we say that each natural number is of the form

$$s(\dots(s(0))\dots) \quad ?$$

In our setting, this means:

Is there a finite sequence of s's followed by a 0 s.t. its interpretation is a natural number.

The size of a natural number.

So this means:

is there an $n \in \mathbb{N}$ s.t. there is a function

$$f : \underbrace{n+1}_{s(n)} \longrightarrow \{s, 0\}$$

s.t. $f(n) = 0$ and $\forall k < n \; f(k) = s$.

Real and complex numbers

If $\mathbb{K} = (K, +, \cdot, 0, 1, <)$ is an ordered field, define

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

where $-$ is additive inverse

Write $K^{\mathbb{N}}$ for the set of fns from $\mathbb{N}$ to K called K-sequences and call a K-seq. t

Cauchy if $\forall \varepsilon\, (\varepsilon > 0 \Rightarrow \exists u\, (u \in \mathbb{N} \wedge \forall m\, (m > u \Rightarrow |t(m) - t(u)| < \varepsilon)))$

null if $\forall \varepsilon\, (\varepsilon > 0 \Rightarrow \exists u\, (u \in \mathbb{N} \wedge \forall m\, (m > u \Rightarrow |t(m)| < \varepsilon)))$

$$CS(K) := \{t \in K^{\mathbb{N}} ;\ t \text{ is Cauchy}\}$$

$t \sim t'$ iff $t - t'$ is null

Thm $CS(K)/\sim$ is an ordered field with the supremum property (every bounded set has a supremum)

We call the elements of this field reals and use the symbol $\mathbb{R}$ for the underlying set.

Remark 1 There are different constructions and it's hard work to show that they are isomorphic.

Remark 2 $\rightsquigarrow \mathbb{C}$.

Augustin-Louis Cauchy



Cauchy around 1840. Lithography by Zéphirin Belliard after a painting by Jean Roller.

Born	21 August 1789 Paris, France
Died	23 May 1857 (aged 67) Sceaux, France
Alma mater	École Nationale des Ponts et Chaussées

§3.4 Replacement

INDUCTION & RECURSION

INDUCTION

N&S: weak

N&S: strong

very bad names: we'll see that "strong" ≠> "weak".

Complete induction

$A \subseteq \mathbb{N}$ if $\emptyset \in A \wedge \forall x\,(x \in A \Rightarrow s(x) \in A)$ then $A = \mathbb{N}$

Order induction

$A \subseteq \mathbb{N}$ if $\forall x\,(\{z \in \mathbb{N};\, z < x\} \subseteq A \Rightarrow x \in A)$ then $A = \mathbb{N}$

In $\mathbb{N}$, they are "equivalent", since they're both true.

Fix $X = (X, <)$ strict partial order. For $x \in X$, let $\mathrm{pred}(x) := \{z \in X;\, z < x\}$.

Def. X satisfies order induction if for all $A \subseteq X$ if $\forall x\,(\mathrm{pred}(x) \subseteq A \Rightarrow x \in A)$ then $A = X$.

Prop. X satisfies order ind. iff $<$ is wellfdd on X

Proof.

NOTE

x is $<$-minimal in $X \setminus A$	iff	$\mathrm{pred}(x) \subseteq A$ $x \notin A$

and apply the definitions. q.e.d.

But now: There is X s.t. it satisfies order induction but not complete induction.

$X := s(\mathbb{N}) = \mathbb{N} \cup \{\mathbb{N}\}$ Clearly wellfounded, but $\mathbb{N} \subsetneq s(\mathbb{N})$.

RECURSION

What is recursion?

Induction is a PROOF PRINCIPLE that allows us to conclude that some $A \subseteq X$ is in fact $A = X$ [on welfdd X].

Recursion is a DEFINITION PRINCIPLE whose some simple conditions uniquely define a function. ← These should be a uniqueness & existence theu.

Example

$$\begin{cases} g : \mathbb{N} \longrightarrow \mathbb{N} \\ h : \mathbb{N}^3 \longrightarrow \mathbb{N} \end{cases}$$

$$\begin{cases} f(n,0) = g(n) \\ f(n,s(k)) = h(n,k,f(n,k)) \end{cases}$$

Warm-up : Think about how to prove uniqueness by induction.

This is precisely the version we use in the Grassmann equations:

Since induction implies recursion (for more details, see Lecture XII)
define by GRASSMANN Recursion Equations

$$n+0 := n \qquad n \cdot 0 := 0$$
$$n+s(m) := s(n+m) \qquad n \cdot s(m) := n \cdot m + n$$

Theorem (Recursion Theorem)

Suppose X is a wellfounded strict partial order.

Let φ be a fmla with free variables x, y, z with the following properties:

functional for all a, b if $\varphi(a,b,c)$ and $\varphi(a,b,c')$, then $c = c'$.

total for all a, b there is a c s.t. $\varphi(a,b,c)$

Then there is a unique F s.t. $\mathrm{dom}(F) = X$ and for all $x \in X$

RECURSION CONDITION (*) $F(x) = z$ iff $\varphi(x, F\restriction \mathrm{pred}(x), z)$.

Proof. Say $Y \subseteq X$ is downwards closed (dwc) if $\forall z, y\ (z < y \wedge y \in Y \Rightarrow z \in Y)$

Call f an **attempt** if $\mathrm{dom}(f)$ is a dwc subset of X and it satisfies (*) on its domain.

Note $\emptyset$ is an attempt and if there is an attempt with $\mathrm{dom}(f) = X$, we have proved existence.

Claim 1 If f, g are attempts and $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ then $f(x) = g(x)$.

[Follows from order induction and (*).]

Claim 1 implies the uniqueness claim of the theorem.