VI Sixth Lecture
LOGIC & SET THEORY

4 February 2026

RECAP OF CHAPTER 1: Propositional Logic

SYNTAX

$\Sigma = P \cup \{\wedge, \vee, \Rightarrow, \neg, \bot, \top,), (\}$ ALPHABET

$Fml \subseteq \Sigma^*$ RECURSIVELY DEFINED

$\rightsquigarrow$ proofs by INDUCTION & definitions by RECURSION

SEMANTICS

Boolean algebras; in particular $\mathbb{2} = (2, +, \cdot, \to, -, 0, 1)$

v valuation : homomorphism from Fml to $\mathbb{2}$

φ valid if $v(\varphi) = 1$ for all valuations

v model of T, $T \models \varphi$, T satisfiable

SYNTAX

T-proof of φ $\rightsquigarrow T \vdash \varphi$

SOUNDNESS : if $T \vdash \varphi$, then $T \models \varphi$.

FINITENESS : if $T \vdash \varphi$, then there is finite $T_0 \subseteq T$ s.t. $T_0 \vdash \varphi$.

T consistent ; Lemma on lemmas, Deduction Theorem

Connecting syntax and semantics

Model Existence Theorem

Every consistent set of formulas is satisfiable.

Compactness : A set is satisfiable if and only if every finite subset is satisfiable.

$\rightsquigarrow$ COMPLETENESS THEOREM

$T \vdash \varphi$ iff $T \models \varphi$.

Proof of the Completeness Theorem:

$T \vdash \varphi$ iff $T \models \varphi$

"$\Rightarrow$" is soundness.

"$\Leftarrow$". Note that if T is inconsistent, both $T \vdash \varphi$ (ex falso) and $T \models \varphi$ (vacuously) are true.

So w.l.o.g. assume T consistent and $T \not\vdash \varphi$.

Then $T \cup \{(\varphi \Rightarrow \bot)\}$ cannot have a model, so it's not satisfiable.

Thus, by MET, it is inconsistent:

$$T \cup \{(\varphi \Rightarrow \bot)\} \vdash \bot$$

By DT: $T \vdash ((\varphi \Rightarrow \bot) \Rightarrow \bot)$

Note that $(((\varphi \Rightarrow \bot) \Rightarrow \bot) \Rightarrow \varphi)$ is an axiom of type 3.

So, by MP, $T \vdash \varphi$. q.e.d.

CHAPTER 2

Predicate Logic (Part I)

a.k.a. FIRST-ORDER LOGIC
a.k.a. QUANTIFIER LOGIC

§ 2.1 Terms & formulas.

Quantifiers $\exists, \forall$ do not exist in prop logic and create an issue with "truth functionality":

$\exists x\, \varphi(x)$

Would like that the truth value of $\exists x \varphi(x)$ can be derived from the truth value of $\varphi(x)$; but $\varphi(x)$ is neither true nor false, but depends on x.

So: idea is to interpret $\varphi(x)$ depending on x and then define the meaning of $\exists x \varphi(x)$.

$\rightsquigarrow$ interpret in a structure!

$\mathfrak{X} = (X, S_0, S_1)$

$S_0 = \{f_i \mid i \in I\}$

$S_1 = \{R_j \mid j \in J\}$

ALPHABET

Fix a signature $a : I \cup \mathcal{F} \to \mathbb{N}$.

Then we take function symbols

NON-LOGICAL SYMBOLS

$$\Sigma_0^a = \{ s_i \mid i \in I \}$$

and relation symbols

$$\Sigma_1^a = \{ r_j \mid j \in \mathcal{J} \},$$

A nullary function symbol is also called a constant symbol.

an infinite set V of variables and logical symbols

$$L := \{ \wedge, \vee, \Rightarrow, \neg, \bot, \top, \exists, \forall,), (\}$$

and call

$$\Sigma^a := \Sigma_0^a \cup \Sigma_1^a \cup V \cup L$$

the alphabet of signature a.

Remark. We do not require $I, \mathcal{J}, V$ to be countable. They could be uncountable.

If they are countable, then so is Σ^a and therefore also $(\Sigma^a)^*$.

TERMS

The set Term^a of a-terms is the smallest subset of $(\Sigma^a)^*$ containing V and closed under the operations

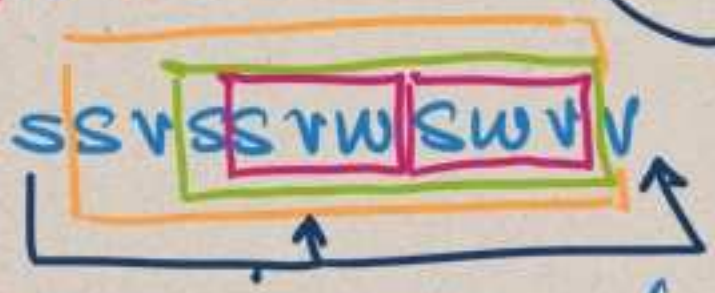
$$t_0, \dots, t_{n-1} \longmapsto s_i t_0 \dots t_{n-1}$$

where $s_i \in \Sigma_0^a$ and $a(i) = n$.

Annoying parentheses

Note that if s is binary and $v, w \in V$, then svw is a term. Not the more readable $s(v,w)$.

This can cause readability issues:



As humans, we would write

$$s(s(v, s(s(v,w), s(w,v))), v)$$

For binary function symbols, we usually use INFIX NOTATION

$$v + w := s(v,w)$$

$$(v + ((v+w) + (w+v))) + v$$

Side remark

Is this even well-defined? Could it be that there are distinct $t_0, \dots, t_{n-1}$ and $t'_0, \dots, t'_{n-1}$ s.t. $s_i t_0 \dots t_{n-1} = s_i t'_0 \dots t'_{n-1}$?

NO $\longrightarrow$ ES#2.

FORMULAS

The set Ful^a of a-formulas is also defined by recursion.

Atomic a-formulas
① $\bot, \top$
② $t = t'$ where $t, t' \in \mathrm{Term}^a$
③ $r^j t_0 \dots, t_{n-1}$ where $r^j \in \Sigma_n^a$ and $a(j) = n$ and $t_i \in \mathrm{Term}^a$.

Ful^a is the smallest subset of $(\Sigma^a)^*$ containing all the atomic a-formulas and closed under

$\varphi, \psi, v \longmapsto$ { $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \Rightarrow \psi)$, $\neg\varphi$, $\exists v\, \varphi$, $\forall v\, \varphi$ }

formulas $\in V$

Remarks (1) The well-definition of $r^j t_0 \dots t_{n-1}$ is a similar issue to the one mentioned on last page ⇝ ES#2.

(2) Since both Term^a and Ful^a are defined as closures, we can prove things about them by induction and define functions on them by recursion.

An important example

$\text{free}(\bot) = \text{free}(\top) := \emptyset$

If t is a term, let $\text{occ}(t)$ be the set of variables occurring in it.

$$
\begin{aligned}
\text{free}(t = t') &:= \text{occ}(t) \cup \text{occ}(t'),\\
\text{free}(r_j t_0 \ldots t_{n-1}) &:= \text{occ}(t_0) \cup \ldots \cup \text{occ}(t_{n-1}),\\
\text{free}((\varphi \wedge \psi)) &:= \text{free}(\varphi) \cup \text{free}(\psi),\\
\text{free}((\varphi \vee \psi)) &:= \text{free}(\varphi) \cup \text{free}(\psi),\\
\text{free}((\varphi \Rightarrow \psi)) &:= \text{free}(\varphi) \cup \text{free}(\psi),\\
\text{free}(\neg\varphi)) &:= \text{free}(\varphi),\\
\text{free}(\exists v \varphi) &:= \text{free}(\varphi) \backslash \{v\}, \text{ and}\\
\text{free}(\forall v \varphi) &:= \text{free}(\varphi) \backslash \{v\}.
\end{aligned}
$$

We say that $\exists, \forall$ bind the variable:

Ex.

$$v = w$$

$$\exists v \quad v = w$$

$$\forall w \exists v \quad v = w$$

This has meaning.

Binding the variables removes the dependency of interpretation on what v, w are.

Def. φ is an **a-sentence** if $\text{free}(\varphi) = \emptyset$.

§ 2.2 TRUTH

Fix a signature a.

Want to interpret a-formulas in a-structures $\mathfrak{X} = (X, S_0, S_1)$.

SYNTAX ⟿ SEMANTICS

$s_i \rightsquigarrow f_i = s_i^{\mathfrak{X}}$

$r_j \rightsquigarrow R_j = r_j^{\mathfrak{X}}$

The truth of formulas will depend on $\mathfrak{X}$:

$\mathbb{N} := (\mathbb{N}, +, 1)$

$\mathbb{Q} := (\mathbb{Q}, +, 1)$

Let a be the relevant signature.
Consider the a-formula

$\varphi := \exists v\,(v + v = 1)$

[Properly, this would be $\exists v\, S_0 v v = S_1$:
S_0 is the symbol for $+$
S_1 is the symbol for 1:
$\exists v S_0 v v = S_1$]

Then
in $\mathbb{N}$, φ is false;
in $\mathbb{Q}$, φ is true.

Next time: DEFINE TRUTH!