

Logic & Set Theory
Lent Term 2026
Part II of the Mathematical Tripos
University of Cambridge
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Example Sheet #2

(14) Let a be any signature. Show that there is no a-term that is a proper initial segment of another a-term. Argue that this implies the unique readibility of the term $s_i t_0 ... t_{n-1}$ and the formula $r_j t_0 ... t_{n-1}$.

(15) Let a be any signature, $\mathbf{X}$ any a-structure, and $\mathfrak{I}$ any $\mathbf{X}$-interpretation. Show

(a) that $\mathbf{X}, \mathfrak{I} \models \exists v(\varphi \vee \psi)$ if and only if $\mathbf{X}, \mathfrak{I} \models (\exists v\varphi \vee \exists v\psi)$ and

(b) that $\mathbf{X}, \mathfrak{I} \models \exists v(\varphi \vee \psi)$ if and only if $\mathbf{X}, \mathfrak{I} \models (\varphi \vee \exists v\psi)$, provided that $v \notin \mathrm{free}(\varphi)$.

Show that the requirement "$v \notin \mathrm{free}(\varphi)$" is necessary in (b).

(16) Let a be any signature. An a-formula is called *negation-free* if it does not contain the symbols $\neg$ and $\bot$. Show that every negation-free a-sentence is satisfiable.

(17) We call a formula φ a Σ_1-*formula* if there is a quantifier free formula ψ and variables $v_1, ..., v_n$ such that $\varphi = \exists v_1 ... \exists v_n \psi$.

Let a be any signature, $\mathbf{X}$ any a-structure, $\mathbf{Y}$ a substructure of $\mathbf{X}$, φ a Σ_1-formula with k free variables, and $\vec{x} \in Y^k$. Show that $\mathbf{Y} \models \varphi(\vec{x})$ implies that $\mathbf{X} \models \varphi(\vec{x})$.

(18) Let a be a signature, a_* be a subsignature of a, $\mathbf{X}$ be an a-structure, and $\mathbf{X}_*$ be its a_*-reduct. Assume that all functions and relations of $\mathbf{X}$ are definable over $\mathbf{X}_*$. Define a surjection $t\colon \mathrm{Fml}^a \to \mathrm{Fml}^{a_*}$ with the property that for any $\mathfrak{I}$ and φ, we have that $\mathbf{X}, \mathfrak{I} \models \varphi$ if and only if $\mathbf{X}_*, \mathfrak{I} \models t(\varphi)$. Justify your claim.

(19) Consider the integers $\mathbb{Z}$ with their standard addition $+$, multiplication $\cdot$, and $0 \in \mathbb{Z}$ and the following three structures: $\mathbf{Z}_1 := (\mathbb{Z}, 0)$, $\mathbf{Z}_2 := (\mathbb{Z}, +, 0)$, and $\mathbf{Z}_3 := (\mathbb{Z}, +, \cdot, 0)$. Fill the following table with "yes" and "no", depending on whether the set A_i is definable over $\mathbf{Z}_j$. Justify your claims.

	$\mathbf{Z}_1$	$\mathbf{Z}_2$	$\mathbf{Z}_3$
$A_1 := \{0\}$			
$A_2 := \{1\}$			
$A_3 := \{z \in \mathbb{Z}\,;\, z \text{ is even}\}$			

(20) Let a be the signature with two binary function symbols $+$ and $\cdot$, a unary function symbol $-$, and two constant symbols 0 and 1. Define a (reasonable) notion of *product* of two a-structures and consider the class of *rings* and the class of *fields*. Write down axiomatisations of both classes. Argue that the former is closed under products, but the latter is not. Explain the difference by analysing the syntactic properties of the two axiomatisations.

(21) Axiomatise the following classes of structures by a set of a-sentences (you are allowed to choose a).

(a) The class of fields of characteristic 2.

(b) The class of partial orders with no maximal element.

(c) The class of algebraically closed fields.

(d) The class of groups of order 60.

(e) The class of simple groups of order 60.

(f) The class of groups that have an element of infinite order.

(g) The class of $\mathbb{R}$-vector spaces.

(22) Can you axiomatise the class in (21f) in the language of groups (i.e., a binary relation symbol $\cdot$, a unary relation symbol $^{-1}$, and a constant symbol 1)?

(23) Let a be the signature with one binary relation symbol $\leq$. Axiomatise the class of dense linear orders with no largest and no smallest element by a set of a-sentences. Show that any two countable models of this theory are isomorphic.

(24) Let a be the signature with two binary function symbols $+$ and $\cdot$ and two constant symbols 0 and 1. Show that the class of fields of positive characteristic is not axiomatisable. Show that the class of fields of characteristic zero is axiomatisable, but not finitely axiomatisable.

(25) Let $\mathbf{X} = (X, E)$ be a structure in the language of set theory. Show that

(a) if the axiom schema of replacement is true in $\mathbf{X}$, then the axiom schema of separation is true in $\mathbf{X}$;

(b) if the axioms of empty set and power set and the axiom schema of replacement are true in $\mathbf{X}$, then the axiom of pairing is true in $\mathbf{X}$.

(c) if for every $x \in X$ there are at most three distinct y such that $y\,E\,x$ and the axioms of extensionality, empty set, and pairing are true in $\mathbf{X}$, then the axiom schema of separation is true in $\mathbf{X}$.

(26) Consider $\mathbf{N} = (\mathbb{N}, <)$ as a structure in the language of set theory. Which of the axioms of Zermelo-Fraenkel set theory are true in $\mathbf{N}$?