Lectio Ultima XVI

Sixteenth & ultimate lecture of FORCING & THE CONTINUUM HYPOTHESIS

Tuesday, 18 March 2025

Forcing	Property	Preservation	Arithmetic
$Fn(\omega\times\aleph_2, 2)$	c.c.c.	all cardinals	$2^{\aleph_0}=\aleph_2$ $2^{\aleph_1}=\aleph_2$
$Fn(\omega\times\aleph_3, 2)$	c.c.c.	all cardinals	$2^{\aleph_0}=\aleph_3$ $2^{\aleph_1}=\aleph_3$ $2^{\aleph_2}=\aleph_3$
$Fn(\aleph_1\times\aleph_3, 2, \aleph_1)$	$\aleph_1$-closed if $M\models CH$, $\aleph_2$-c.c.	$\aleph_1$ preserved CLOSURE LEMMA [not yet proved] $\kappa\geq\aleph_2$ preserved	if $M\models CH$ $2^{\aleph_0}=\aleph_1$ $2^{\aleph_1}=\aleph_3$

more on this later (p. 4)

pp. 2&3

PSA $\forall\alpha<\beta\,(2^{\aleph_\alpha}<2^{\aleph_\beta})$

Note: GCH $\longrightarrow$ PSA, but our model of $\neg$CH fails PSA.

Q Can we have $\aleph_1<2^{\aleph_0}<2^{\aleph_1}$?

CLOSURE LEMMA

Theorem If $\mathbb{P}$ is λ-closed and $\kappa < \lambda$, then $\wp(\kappa) \cap M = \wp(\kappa) \cap M[G]$.

To be proved in Lecture XVI.

Corollary Forcing with $\mathrm{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1)$
(a) does not change $\wp(\omega)$
(b) therefore preserves $\aleph_1$
[see ES#1 and the relation between codes for ctble wellorders and preserving $\aleph_1$]

λ-closed: every descending seq. of length $< \lambda$ has a lower bound

Proof. Let $f \in M[G]$, $f : \kappa \longrightarrow 2$ and assume towards contradiction that $f \notin M$. $\longrightarrow f \notin \mathbb{B}$.
$\mathbb{B} := \{f \in M ; f : \kappa \longrightarrow 2\}$ Let τ be a name for f.
By Forcing Theorem, there is $p \in G$ s.t. $p \Vdash \tau : \check{\kappa} \longrightarrow \check{2} \wedge \tau \notin \check{\mathbb{B}}$.
Construct a κ-sequence of conditions p_α, $\alpha \leq \kappa$. [DESCENDING]
$p_0 := p$.
If α is limit $\{p_\beta ; \beta < \alpha\}$ is a descending seq. so by λ-closure, let p_α be a lower bound.
[Includes $\alpha = \kappa$.]
If p_α is defined, $p_\alpha \leq p$, so $p_\alpha \Vdash \tau : \check{\kappa} \longrightarrow \check{2}$.
In particular $p_\alpha \Vdash \tau(\check{\alpha}) = \check{0} \vee \tau(\check{\alpha}) = \check{1}$.
By def of $\Vdash \neg\varphi$, we find $q \leq p_\alpha$ s.t. either $q \Vdash \tau(\check{\alpha}) = \check{0}$ or $q \Vdash \tau(\check{\alpha}) = \check{1}$.

Let $p_{\alpha+1} := q$.

The sequence $\{p_\alpha ; \alpha \leq \kappa\}$ is defined in M [by Definability Thm], so we can define

$$g(\alpha) = 1 :\iff p_{\alpha+1} \Vdash \tau(\check{\alpha}) = \check{1}.$$

Then $g \in M$.

But now
$$p_\kappa \Vdash \tau(\check{\alpha}) = \check{1} \quad \text{or} \quad p_\kappa \Vdash \tau(\check{\alpha}) = \check{0}$$
for all α.

So $p_\kappa \Vdash \tau = \check{g}$.

$\implies p_\kappa \Vdash \tau \in \check{B}$.

But $p_\kappa \leq p \Vdash \tau \notin \check{B}$. Contradiction!

q.e.d.

IMPORTANT REMARK

Check notes of L XV very carefully!

Lecture $\overline{XV}$, page 7:

With example (38), we need to figure out the c.c. of $Fn(\aleph_1 \times \aleph_3, 2, \aleph_1)$.
We need a more general ΔSL for this.

If A is a family of sets of size $< \kappa$ such that $|A| = \kappa^+$ & $\kappa^{<\kappa} = \kappa$ then there is a ΔS $D \subseteq A$ s.t. $|D| = \kappa^+$.

Kunen's book.

EXAMPLE SHEET #3.

(38) If κ is a cardinal, we say that $\mathbb{P}$ has the κ-c.c. if every antichain in $\mathbb{P}$ has cardinality smaller than κ. (Thus, the c.c.c. is the $\aleph_1$-c.c.) If κ is a cardinal in M, we say that $\mathbb{P}$ *preserves cardinals* $\geq \kappa$ if for every $\mathbb{P}$-generic filter G over M and every $\lambda \geq \kappa$, we have that $M \models$ "λ is a cardinal" if and only if $M[G] \models$ "λ is a cardinal". Show that if $M \models$ "κ is a regular cardinal" and $M \models$ "$\mathbb{P}$ has the κ-c.c.", then $\mathbb{P}$ preserves cardinals $\geq \kappa$.

This ΔSL gives with same proof as before:
If $M \models 2^{\aleph_1} = \aleph_2$, then
$Fn(\aleph_1 \times \aleph_3, 2, \aleph_1)$ has the $\aleph_2^M$-c.c.

So: Forcing over a model of GCH [or at least $2^{\aleph_1} = \aleph_2$]
$Fn(\aleph_1 \times \aleph_3, 2, \aleph_1)$ preserves cardinals $\geq \aleph_2$.

Note that while $Fn(X, Y, \aleph_1)$ is always $\aleph_1$-closed, the chain condition depended on the value of $\aleph_1^{\aleph_0}$. The partial order

$$Fn(\aleph_1 \times \aleph_3, 2, \aleph_1) \text{ has in general}$$

the $(2^{\aleph_0})^+$-chain condition.

CH $\Longrightarrow (2^{\aleph_0})^+ = \aleph_2$, so all cardinals are preserved.

However, if $2^{\aleph_0} > \aleph_1$, then there is a gap and we do not know whether cardinals are preserved.

If $M \models 2^{\aleph_0} = \aleph_2$, does $\mathrm{Fn}(\aleph_1^M \times \aleph_3^M, 2, \aleph_1^M)$ preserve $\aleph_2^M$?

<u>Answer</u>

($\circledast$) $\mathbb{P} := \mathrm{Fn}(\lambda^+ \times \kappa, 2, \lambda^+)$ ALWAYS ADDS a surjection from λ^+ to $2^\lambda \cap M$

<u>Application</u> If $\lambda = \aleph_0$ and $M = 2^{\aleph_0} = \aleph_2$, then $\mathrm{Fn}(\aleph_1^M \times \kappa, 2, \aleph_1^M)$ adds a surjection from $\aleph_1^M$ onto $\wp(\aleph_0) \cap M$, – i.e., $\aleph_2^M$. So $|\aleph_2^M| = \aleph_1^{M[G]}$

!!!

<u>Proof of ($\circledast$)</u> A generic for $\mathbb{P}$ "is" a map

$$f : \lambda^+ \times \kappa \longrightarrow 2$$

Define $h : \lambda^+ \longrightarrow 2^\lambda$ by

$$h(\alpha)(\beta) = 1 \;:\Longleftrightarrow f(\alpha, \beta) = 1.$$

Claim: h is a surjection onto $2^\lambda \cap M$.

If $g \in M$, $g : \lambda \longrightarrow 2$, consider

$$D_g := \{ p \,;\, \exists \alpha < \lambda^+ \,\forall \beta < \lambda \;\; p(\alpha, \beta) = 1 \Longleftrightarrow g(\beta) = 1 \}$$

This is dense, and thus $g \in \mathrm{ran}(h)$.

q.e.d.

Back to our question

Can we get $\aleph_1 < 2^{\aleph_0} < 2^{\aleph_1}$?

Start with $M \models GCH$.
Consider $\mathbb{P} := Fn(\aleph_1 \times \aleph_3, 2, \aleph_1) \cap M$
and let G be $\mathbb{P}$-generic$/M$.
Consider $\mathbb{Q} := Fn(\omega \times \aleph_2, 2) \cap M[G]$.
and let H be $\mathbb{Q}$-generic$/M[G]$.

Claim : $M[G][H] \models \aleph_1 < 2^{\aleph_0} < 2^{\aleph_1}$.

FORCING ITERATION "two-step iteration"

1. $\mathbb{P}$ is card. preserving over M since $M \models GCH$:
   $\Longrightarrow \aleph_\alpha^M = \aleph_\alpha^{M[G]}$.
2. $\mathbb{Q}$ has c.c.c., so is card. preserving:
   $\Longrightarrow \aleph_\alpha^{M[G][H]} = \aleph_\alpha^{M[G]} = \aleph_\alpha^M$.
3. $M[G] \models 2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_3$.
   [Lecture XV; since $M \models CH$]
4. Then $M[G][H] \models 2^{\aleph_0} = \aleph_2 \wedge 2^{\aleph_1} \geq \aleph_3$
   [using a nice name analysis, we could calculate $2^{\aleph_1} = \aleph_3$].

This proves the claim.

IMPORTANT: The order of forcings matters!

Suppose $M \models GCH$

$$\mathbb{Q}' := \mathrm{Fn}(\omega \times \aleph_2, 2) \cap M$$

$$H \quad \mathbb{Q}'\text{-gen.}/M$$

$$\mathbb{P}' := \mathrm{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1) \cap M[H]$$

$$G \quad \mathbb{P}'\text{-gen.}/M[H].$$

Consider $M[H][G]$, then

$$M[H] \models 2^{\aleph_0} = \aleph_2 = 2^{\aleph_1}.$$

But that means that forcing with $\mathbb{P}'$ **WILL** collapse $\aleph_2^M = \aleph_2^{M[H]}$.

Since $\mathbb{P}'$ is $\aleph_1$-closed,

$$\mathcal{P}(\omega) \cap M[H] = \mathcal{P}(\omega) \cap M[H][G]$$

In particular, $M[H][G] \models CH$.

So, this order does not achieve what we want.

Final Remarks on forcing CH:

We have already seen (p. 7) that we can ACCIDENTALLY force CH: can we also do it on purpose?

Assume $M \models 2^{\aleph_0} = \aleph_2$.

<u>Q</u>: Can you obtain $M[G] \models CH$?

The natural forcing would be

$$\mathbb{P} := Fn(\omega, \aleph_1^M).$$

This collapses $\aleph_1^M$; it does not have c.c.c., but since it has size $\aleph_1^M$, it has the $\aleph_2^M$-c.c., so all cardinals $\geq \aleph_2^M$ are preserved.

Clearly therefore:

$$M[G] \models |\mathcal{P}(\omega) \cap M| = \aleph_2^M = \aleph_1^{M[G]}.$$

But: is $|\mathcal{P}(\omega) \cap M| = |\mathcal{P}(\omega) \cap M[G]|$?

Analysis of nice names:
- size $\aleph_1^M$
- size of antichains $\leq \aleph_1^M$

So: upper bound

$$(\aleph_1^M)^{\aleph_1^M \cdot \aleph_0} = (2^{\aleph_1^M})^M$$

Note that if $f : \aleph_1 \longrightarrow 2$, $f \in M$ and $h : \omega \longrightarrow \aleph_1^M$ is a bijection with $h \in M[G]$. Then $f \circ h : \omega \longrightarrow 2$ in $M[G]$ and for $f \neq g$, $f \circ h \neq g \circ h$, so every such fn f in M becomes a new subset of ω in $M[G]$.

Thus: only if $M \models 2^{\aleph_1} = \aleph_2$ can we conclude that $M[G] \models CH$.