

# Lectio Ultima XVI

Sixteenth & ultimate lecture of FORCING &  
THE CONTINUUM HYPOTHESIS

Tuesday, 18 March 2025

Forcing	Property	Preservation	Ambiguity
$\text{Tu}(\omega \times \aleph_2, 2)$	c.c.c.	all cardinals	$2^{\aleph_0} = \aleph_2$ $2^{\aleph_1} = \aleph_2$
$\text{Tu}(\omega \times \aleph_3, 2)$	c.c.c.	all cardinals	$2^{\aleph_0} = \aleph_3$ $2^{\aleph_1} = \aleph_3$ $2^{\aleph_2} = \aleph_3$
$\text{Tu}(\aleph_1 \times \aleph_3, 2^{\aleph_1})$	$\aleph_1$ -closed  if $M \models \text{CH}$ , $\aleph_2$ -c.c.	$\aleph_1$ preserved CLOSURE LEMMA [not yet proved]  $\kappa \geq \aleph_2$ preserved	$2^{\aleph_0} = \aleph_1$ $2^{\aleph_1} = \aleph_3$

wire on this  
later (p. 4)

PSA

$$\forall \alpha < \beta (2^{\aleph_\alpha} < 2^{\aleph_\beta})$$

pp. 283

Note:  $\text{GCH} \rightarrow \text{PSA}$ , but our model of  
 $\neg \text{CH}$  fails PSA.

Q

Can we have  $\aleph_1 < 2^{\aleph_0} < 2^{\aleph_2}$ ?

# CLOSURE LEMMA

Theorem: If  $\dot{P}$  is  $\lambda$ -closed and  $\kappa < \lambda$ , then  $\dot{P}(\kappa) \cap M = P(\kappa) \cap M[G]$ .

Corollary: Forcing with  $\text{Fn}(\lambda_1 \times \lambda_3, 2, \lambda_1)$

- (a) does not change  $P(\omega)$
- (b) therefore preserves  $\lambda_1$   
 [see ES#1 and relationship between codes for able wellorders and preserving  $\lambda_1$ ]

To be proved in Lecture XVI.

$\lambda$ -closed: every descending seq. of length  $< \lambda$  has a lower bound

Proof: Let  $f \in M[\mathbb{Q}]$ ,  $f: \kappa \rightarrow 2$  and assume towards contradiction that  $f \notin M$ .  $\rightarrow p \notin B$ .

$B := \{f \in M; f: \kappa \rightarrow 2\}$  Let  $\tau$  be a name for  $f$ .

By Forcing Theorem, there is  $g \in G$  s.t.  $p \Vdash \tau : \check{\kappa} \rightarrow 2^{\check{\lambda}}$

Construct a  $\kappa$ -sequence of conditions  $p_\alpha$ ,  $\alpha \leq \kappa$ .

$p_0 := p$ .

[DESCENDING]

If  $\alpha$  is limit  $\{p_\beta; \beta < \alpha\}$  is a descending seq.

[Includes  $\alpha = \kappa$ .] so by  $\lambda$ -closure, let  $p_\alpha$  be a lower bound.

If  $p_\alpha$  is defined,  $p_\alpha \leq p$ , so  $p_\alpha \Vdash \tau : \check{\kappa} \rightarrow 2$ .

In particular  $p_\alpha \Vdash \tau(\check{\alpha}) = 0 \vee \tau(\check{\alpha}) = 1$ .

By def of  $\vdash \neg \varphi$ , we find  $q \leq p_\alpha$  s.t.

either  $q \Vdash \tau(\check{\alpha}) = 0$  or  $q \Vdash \tau(\check{\alpha}) = 1$ .

Let  $p_{\alpha+1} := q$ .

The sequence  $\{p_\alpha; \alpha \leq k\}$  is defined in  $M$  [by Definability Tree], so we can define

$$g(\alpha) = 1 : \iff p_{\alpha+1} \Vdash \tau(\check{\alpha}) = \check{1}.$$

Then  $\bigvee g \in M$ .

But now  $p_k \Vdash \tau(\check{\alpha}) = \check{1}$  or  $p_k \Vdash \tau(\check{\alpha}) = \check{0}$  for all  $\alpha$ .

So  $p_k \Vdash \tau = \bigvee^v g$ .  
 $\implies p_k \Vdash \tau \in \mathcal{B}^v$ .

But  $p_k \leq p \Vdash \tau \notin \mathcal{B}$ . Contradiction!

q.e.d.

# IMPORTANT REMARK

Check value of  
L<sub>XI</sub> very carefully!

Lecture XV,  
page 7:

With example (38), we need to figure out the c.c. of  $\text{Tu}(\aleph_1 \times \aleph_3, 2, \aleph_1)$ . We need a more general DSL for this.

If  $A$  is a family of sets of  $\kappa^{+} < \kappa$  such that  $|A| = \kappa^{+}$  &  $\kappa^{+} = \kappa$ , then  $\text{Tu}(A)$  is a DS DGA s.t.  $|\text{D}| = \kappa^{+}$ .

## EXAMPLE SHEET #3.

- (38) If  $\kappa$  is a cardinal, we say that  $\mathbb{P}$  has the  $\kappa$ -c.c. if every antichain in  $\mathbb{P}$  has cardinality smaller than  $\kappa$ . (Thus, the c.c. is the  $\aleph_1$ -c.c.) If  $\kappa$  is a cardinal in  $M$ , we say that  $\mathbb{P}$  preserves cardinals  $\geq \kappa$  if for every  $\mathbb{P}$ -generic filter  $G$  over  $M$  and every  $\lambda \geq \kappa$ , we have that  $M \models \lambda$  is a cardinal if and only if  $M[G] \models \lambda$  is a cardinal". Show that if  $M \models \kappa$  is a regular cardinal" and  $M \models \mathbb{P}$  has the  $\kappa$ -c.c.", then  $\mathbb{P}$  preserves cardinals  $\geq \kappa$ .

This DSL gives with some proof as before:

If  $M \models 2^{\aleph_0} = \aleph_2$ , then

$\text{Tu}(\aleph_1 \times \aleph_3, 2, \aleph_1)$  has the  $\aleph_2^{\aleph_0}$ -c.c.

So. Forcing over a model of GCH [or at least  $2^{\aleph_0} = \aleph_2$ ]  $\text{Tu}(\aleph_1 \times \aleph_3, 2, \aleph_1)$  preserves cardinals  $\geq \aleph_2$ .

Note that while  $\text{Tu}(\kappa, \lambda, \mu)$  is always  $\lambda$ -closed, the chain condition depends on the value of  $\lambda^{\aleph_0}$ . The partial order

$\text{Tu}(\aleph_1 \times \aleph_3, 2, \aleph_1)$  has in general the  $(2^{\aleph_0})^+$ -chain condition.

CH  $\rightarrow (2^{\aleph_0})^+ = \aleph_2$ , so all cardinals are preserved.

However, if  $2^{\aleph_0} > \aleph_2$ , then there is a gap and we do not know whether cardinals are preserved.

Kunen's book.

1.6. LEMMA. Let  $\kappa$  be any infinite cardinal. Let  $\theta > \kappa$  be regular and satisfy  $\lambda \in \theta \cap \kappa^{+} < \theta$ , assume  $|\lambda| \leq \theta$  and  $\lambda \neq \kappa$ . If  $\kappa \in \lambda \cap \theta \cap \kappa^{+}$ , then there is  $\alpha \in \lambda$  such that  $|\alpha| = \theta$  and  $\alpha$  forms a  $\lambda$ -chain.

PROOF. By shrinking  $\theta$  if necessary, we may assume  $|\lambda| = \theta$ . Then  $\lambda \cap \kappa^{+} \neq \emptyset$ . Since when the elements of  $\lambda$  are its individuals, we may assume  $\lambda \cap \kappa^{+} \neq \emptyset$ . Then each  $\kappa \in \lambda$  has some order type  $\kappa$  as a subset of  $\theta$ . Since  $\theta$  is regular and  $\theta > \kappa$ , there is some  $\beta < \theta$  such that  $\lambda \cap \kappa^{+}$  has type  $\beta$  has cardinality  $\theta$ . We now fix such a  $\beta$  and deal only with it.

For each  $\kappa < \theta$ ,  $\kappa^{+} < \theta$  implies that less than  $\theta$  elements of  $\lambda \cap \kappa^{+}$  are subsets of  $\kappa$ . Thus,  $\lambda \cap \kappa^{+}$  is unbounded in  $\theta$ . If  $\kappa < \theta$ , and  $\zeta < \beta$ , let  $\zeta\beta$  be the  $\beta$ -th element of  $\kappa$ . Since  $\theta$  is regular, there is some  $\gamma$  such that  $\lambda \cap \kappa^{+} \cap \zeta\beta$  is unbounded in  $\theta$ . Now let  $\zeta_0$  be the least such  $\zeta$  for which  $\lambda \cap \kappa^{+} \cap \zeta\beta$  is bounded in  $\theta$ . Then  $\zeta_0 < \theta$  and  $\zeta_0\beta < \zeta_0$  for all  $\kappa < \zeta_0$ , and all  $\kappa < \zeta_0$ .



Figure 1.1. A  $\lambda$ -chain

9.10. LEMMA.  $\text{Tu}(\kappa, \lambda, \mu)$  has the  $(\lambda)^{+\aleph_0}$ -c.c.

PROOF. Let  $\theta = (\lambda)^{+\aleph_0}$ , and suppose that  $\langle p_\alpha | \alpha < \theta \rangle$  formed an anti-chain. Fix, assume  $\theta$  is regular. Then  $(\lambda)^{+\aleph_0} = |\lambda|^{\aleph_0} \leq \theta^{\aleph_0} < \theta$ , so by the  $\lambda$ -system lemma (see 1.6) there is an  $\lambda \subset \theta$  with  $|\lambda| = \theta$  such that  $\{\text{dom } p_\alpha | \alpha \in \lambda\}$  forms a  $\lambda$ -chain with some root  $\kappa$ . Since there are less than  $\theta$  possibilities for  $p_\alpha[\kappa]$ , we have a contradiction as in the proof for  $\kappa = \omega$  (see Lemma 1.4).

If  $\lambda$  is singular, then since  $\theta$  is regular and  $\lambda < \theta$ , we could find a regular  $\kappa < \lambda$  such that  $\lambda = \text{dom } p_\alpha \cap \kappa$  has cardinality  $\theta$ . Then  $\langle p_\alpha | \alpha < \theta \rangle$  contradicts the  $(\lambda)^{+\aleph_0}$ -c.c., which we have just proved for regular  $\lambda$ .  $\square$

If  $M \models 2^{\lambda^0} = \lambda_2$ , does  
 $\text{Tu}(\lambda_1^M \times \lambda_3^M, 2, \lambda_1^M)$  preserve  $\lambda_2^M$ ?

Answer

$\text{P} := \text{Tu}(\lambda^+ \times \kappa, 2, \lambda^+)$  ALWAYS ADDS  
 a surjection from  $\lambda^+$  to  $2^{\lambda^0 M}$

Application If  $\lambda = \lambda^0$  and  $M = 2^{\lambda^0} = \lambda_2$ , then  
 $\text{Tu}(\lambda_1^M \times \kappa, 2, \lambda_1^M)$  adds a surjection  
 from  $\lambda_1^M$  onto  $P(\lambda^0) \cap M$ , i.e.,  
 $\lambda_2^M$ . So  $|\lambda_2^M| = |\lambda_1^M|$  !!!

Proof of  $\textcircled{1}$  A generic for  $\text{P}$  is a map  
 $f: \lambda^+ \times \kappa \rightarrow 2$

Define  $h: \lambda^+ \rightarrow 2^\lambda$  by

$$h(\alpha)(\beta) = 1 : \iff f(\alpha, \beta) = 1.$$

Claim:  $h$  is a surjection onto  $2^{\lambda^0 M}$ .

If  $g \in M$ ,  $\langle g: \lambda \rightarrow 2 \rangle$ , consider

$$\begin{aligned} \exists g &= \lambda p; \exists \alpha < \lambda^+ \forall \beta < \lambda \quad p(\alpha, \beta) = 1 \\ &\iff g(\beta) = 1 \end{aligned}$$

This is dense, and thus  $g \in \text{ran}(h)$ . q.e.d.

Back to our question

Can we get  $\aleph_1 < 2^{\aleph_0} < 2^{\aleph_1}$ ?

Start with  $M \models \text{GCH}$ .

Consider  $P := \text{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1) \cap M$   
and let  $G$  be  $P$ -generic  $| M$ .

Consider  $Q := \text{Fn}(\omega \times \aleph_2, 2) \cap M[G]$ .  
and let  $H$  be  $Q$ -generic  $| M[G]$ .

Claim :  $M[G][H] \models \aleph_1 < 2^{\aleph_0} < 2^{\aleph_1}$ .  
**FORCING ITERATION** "two-step iteration"

1.  $P$  is c.c.d. preserving over  $M$  since  $M \models \text{GCH}$ :  
 $\rightarrow \aleph_u^M = \aleph_u^{M[G]}$ .

2.  $Q$  has c.c.c., so is c.c.d. preserving:  
 $\rightarrow \aleph_u^{M[G][H]} = \aleph_u^{M[G]} = \aleph_u^M$ .

3.  $M[G] \models 2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_3$ .

[Lecture XV; since  $M \models \text{CH}$ ]

4. Then  $M[G][H] \models 2^{\aleph_0} = \aleph_2 \wedge 2^{\aleph_1} \geq \aleph_3$   
[using a nice name analysis, we could calculate  $2^{\aleph_1} = \aleph_3$ ].

This proves the claim.

IMPORTANT : The order of forcing matters!

Suppose  $M \models QCH$

$$Q' := \text{Fn}(\omega \times \aleph_2, 2) \cap M$$

H  $Q'$ -fin. / M

$$P' := \text{Fn}(\aleph_1 \times \aleph_3, 2, \alpha_1) \cap M^{[+]}$$

G  $P'$ -fin. /  $M^{[+]}$ .

Consider  $M^{[+]}\llbracket G \rrbracket$ , then

$$M^{[+]} \models 2^{\aleph_0} = \aleph_2 = 2^{\aleph_1}.$$

But that means that forcing with  $P'$  will collapse  $\aleph_2^{M^{[+]}} = \aleph_2^{M^{[+]}}$ .

Since  $P'$  is  $\aleph_1$ -closed,

$$\beta(\omega) \cap M^{[+]} = \beta(\omega) \cap M^{[+]}\llbracket G \rrbracket$$

In particular,  $M^{[+]}\llbracket G \rrbracket \models CH$ .

So, this order does not achieve what we want.

## Final Remarks on forcing CH:

Assume  $M \models 2^{\aleph_0} = \aleph_2$ .

We have already seen (p. 7) that we can ACCIDENTALLY force CH: can we also do it on purpose?

Q: Can you obtain  $M[G] \models \text{CH}^2$ ?

The natural forcing would be

$$P := \text{Fn}(\omega, \aleph_1^M)$$

This collapses  $\aleph_1^M$ ; it does not have c.c.c., but since it has size  $\aleph_1^M$ , it has the  $\aleph_2^M$ -c.c., so all cardinals  $\geq \aleph_2^M$  are preserved.

Clearly therefore:

$$M[G] \models |\rho(\omega) \cap M| = \aleph_2^M = \aleph_1^{M[G]}$$

But: is  $|\rho(\omega) \cap M| = |\rho(\omega) \cap M[G]|$ ?

Analysis of size issues:

- size  $\aleph_1^M$

- size of antichains  $\leq \aleph_1^M$

∴ upper bound

$$(\aleph_1^M)^{\aleph_1^M \cdot \aleph_0} = (2^{\aleph_1^M})^M$$

Note that if  $f: \aleph_1 \rightarrow 2$ ,  $f \in M$  and  $h: \omega \rightarrow \aleph_1^M$  is a bijection with  $h \in M[G]$ . Then  $f \circ h: \omega \rightarrow 2$  is in  $M[G]$  and for  $f \neq g$ ,  $f \circ h \neq g \circ h$  so every such  $f \circ h$  in  $M$  becomes a new subset of  $\omega$  in  $M[G]$ . Thus: only if  $M \models 2^{\aleph_1} = \aleph_2$  can we conclude that  $M[G] \models \text{CH}$ .