XV

Fifteenth & Penultimate
Lecture of
FORCING & THE CONTINUUM HYPOTHESIS
Saturday, 15 March 2025

Lecture XIII
1. Y countable $\Rightarrow$ $\mathrm{Fn}(X,Y)$ c.c.c.
2. $\mathbb{P}$ ccc $\Rightarrow$ $\mathbb{P}$ preserves cardinals
3. $\mathrm{Fn}(\omega \times \aleph_\alpha^M, 2)$ forces $2^{\aleph_0} \geq \aleph_\alpha^M$.

Lecture XIV Technique of "nice names"
If $\mathbb{P}$ ccc and size κ, then
$\mathbb{P}$ forces $2^{\aleph_0} \leq (\kappa^{\aleph_0})^M$.

Thus: Models with $2^{\aleph_0} = \aleph_\alpha$ for all $\alpha > 0$ with $\mathrm{cf}(\alpha) > \aleph_0$.

This is the final answer to the question of CH.

! IN ZFC ! $\rightsquigarrow$ GÖDEL'S PROGRAMME
Quest for additional axioms A s.t.
$\mathrm{ZFC} + A \vdash \mathrm{CH}$ or
$\mathrm{ZFC} + A \vdash \neg \mathrm{CH}$.

A different question:
What about GCH, i.e., other values of $2^{\aleph_\alpha}$ $\alpha > 0$?

What about $2^{\aleph_1}$?

More on nice names:

Generalize "nice names" to **λ-nice names**:

Let $\mathcal{A} = \{A_\alpha ; \alpha < \lambda\}$ be a family of λ many maximal $\mathbb{P}$-antichains:

$$\tau_{\mathcal{A}} := \{(\check{\alpha}, p) ; p \in A_\alpha\}$$
for $\alpha \in \lambda$.

These are names for subsets of λ.

Observe that our theorem "every $A \subseteq \lambda$ in $M[G]$ has a λ-nice name" still goes through.

If κ is an M-cardinal s.t. every antichain of $\mathbb{P}$ has size $\leq \kappa$. [On ES#3, this is called the κ^+-chain condition.]

Let $\mu := |\mathbb{P}|$ (in M).

Then $(\mu^\kappa)^\lambda = \mu^{\kappa \cdot \lambda}$ is an upper bound on the number of λ-nice names.

Thus $M[G] \models 2^\lambda \leq (\mu^{\kappa \cdot \lambda})^M$.

Q. Forcing with $\mathbb{P} := \mathrm{Fn}(\omega \times \aleph_2, 2)$ and calculate $2^{\aleph_1}$.

Assume $M \models GCH$. Then:

$$\mu = |\mathbb{P}| = \aleph_2$$
$$\lambda = \aleph_1$$
$$\kappa = \aleph_0 \quad [\text{since } \mathbb{P} \text{ has c.c.c.}]$$

$$\Longrightarrow M[G] \models 2^{\lambda} \leq \aleph_2^{\aleph_1 \cdot \aleph_0} = (\aleph_2^{\aleph_1})^M.$$

Calculate $(\aleph_2^{\aleph_1})^M$:

$$\aleph_2^{\aleph_1} \underset{\substack{\text{Hausdorff's}\\ \text{Formula}}}{=} \aleph_2 \cdot 2^{\aleph_1} \underset{GCH}{=} \aleph_2 \cdot \aleph_2 = \aleph_2.$$

Together: $M[G] \models \boxed{2^{\aleph_1} = \aleph_2 = 2^{\aleph_0}}.$

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Is the Dream Solution of the Continuum Hypothesis Attainable?

Joel David Hamkins

Let me turn now to a second illustration of this pattern of response. Consider the set-theoretic principle that I have called the *powerset size axiom* (PSA), the axiom asserting plainly that smaller sets have fewer subsets:

$$\forall x, y, \quad |x| < |y| \Rightarrow \big|P(x)\big| < \big|P(y)\big|.$$

Set-theorists understand the situation of this axiom very well, and I shall shortly explain. But how is it received in mathematics generally? Extremely well! A large number of mathematicians, including some very good ones (invariably from non-logic-related areas of mathematics), look favorably upon the axiom when it is first considered, viewing it as highly natural or even obviously true. They take the axiom to express what seems be a basic intuitive principle, namely, a strictly smaller set should have strictly fewer subsets. The principle, for example, is currently the top-rated answer (see Hamkins [4]) among dozens to a popular MathOverflow question seeking examples of reasonable-sounding statements that are nevertheless independent of the axioms of set theory, and the same issue has arisen in at least three other MathOverflow questions, posted by mathematicians asking naively whether the PSA is true, or how to prove it or indeed asking with incredulity how it could not be provable. My experience is that a brief conversation with mathematicians at your favorite math tea stands a good chance to turn up additional examples of mathematicians who find the axiom to express a basic set-theoretic fact.

Q. Is it possible to get PSA, i.e.,

$$\forall \kappa, \lambda \quad \kappa < \lambda \longrightarrow 2^{\kappa} < 2^{\lambda}$$

without CH.

In particolo, can we get

$$2^{\aleph_0} = \aleph_2$$
$$2^{\aleph_1} = \aleph_3 .$$

?

First idea

Force with $Fn(\aleph_1 \times \aleph_3, 2)$:

1. Yields $\aleph_3$ many subsets of $\aleph_1$.
2. Still has ccc, so all cardinals are preserved
3. How many $\aleph_1$-nice names are there:

$$\aleph_3^{\aleph_1 \cdot \aleph_0} = \aleph_3^{\aleph_1} \underset{\text{Hausdorff}}{=} \aleph_3 \cdot 2^{\aleph_1} \underset{\text{GCH}}{=} \aleph_3 .$$

Get: $2^{\aleph_1} = \aleph_3$ in $M[G]$.

Unfortunately, $M[G] \models 2^{\aleph_0} = \aleph_3$.

Interpret the generic object G as

$$f_\alpha : \aleph_1 \longrightarrow 2 \quad \text{for } \alpha < \aleph_3 .$$

Define $g_\alpha := f_\alpha \upharpoonright \omega$.

Claim: For $\alpha \neq \alpha'$, $g_\alpha \neq g_{\alpha'}$.

This is since

$$D_{\alpha,\alpha'} := \{ p \,;\, \exists n \in \omega \ g_\alpha(n) \neq g_{\alpha'}(n) \}$$

is still dense, so $g_\alpha \neq g_{\alpha'}$.

So, forcing with $Fn(\aleph_1 \times \aleph_3, 2)$ gives the same situation as forcing with $Fn(\omega \times \aleph_3, 2)$ for $2^{\aleph_0}, 2^{\aleph_1}$.

Idea

$$\mathrm{Fn}(X, Y, \kappa) := \{ p \,;\, \mathrm{dom}(p) \subseteq X, \mathrm{ran}(p) \subseteq Y, |p| < \kappa \}$$

Thus $\mathrm{Fn}(X, Y) = \mathrm{Fn}(X, Y, \aleph_0)$.

Consider $\mathbb{P} := \mathrm{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1)$

Properties
1. We still have $M[G] \models 2^{\aleph_1} \geq \aleph_3^M$
2. Not clear that this forcing is preserving cardinals!

First goal:

What about preserving cardinals?

Clearly, $\mathrm{Fn}(\aleph_1 \times \aleph_3, 2, \aleph_1)$ does not have the c.c.c. anymore.

With example (38), we need to figure out the c.c. of $Fn(\aleph_1 \times \aleph_3, 2, \aleph_1)$.
We need a more general ΔSL for this:

If A is a family of sets of size $< \kappa$ such that $|A| = \kappa^+$ & $\kappa^{<\kappa} = \kappa$ then there is a ΔS $D \subseteq A$ s.t. $|D| = \kappa^+$.

<u>Kunen's book.</u>

1.6. THEOREM. *Let κ be any infinite cardinal. Let $\theta > \kappa$ be regular and satisfy $\forall \alpha < \theta (|\alpha^{<\kappa}| < \theta)$. Assume $|\mathscr{A}| \geq \theta$ and $\forall x \in \mathscr{A} (|x| < \kappa)$, then there is a $\mathscr{B} \subset \mathscr{A}$, such that $|\mathscr{B}| = \theta$ and $\mathscr{B}$ forms a Δ-system.*

PROOF. By shrinking $\mathscr{A}$ if necessary, we may assume $|\mathscr{A}| = \theta$. Then $|\bigcup \mathscr{A}| \leq \theta$. Since what the elements of $\mathscr{A}$ are as individuals is irrelevant, we may assume $\bigcup \mathscr{A} \subseteq \theta$. Then each $x \in \mathscr{A}$ has some order type $< \kappa$ as a subset of θ. Since θ is regular and $\theta > \kappa$, there is some $\rho < \kappa$, such that $\mathscr{A}_1 = \{x \in \mathscr{A} : x \text{ has type } \rho\}$ has cardinality θ. We now fix such a ρ and deal only with $\mathscr{A}_1$.

For each $\alpha < \theta$, $|\alpha^{<\kappa}| < \theta$ implies that less than θ elements of $\mathscr{A}_1$ are subsets of α. Thus, $\bigcup \mathscr{A}_1$ is unbounded in θ. If $x \in \mathscr{A}_1$ and $\xi < \rho$, let $x(\xi)$ be the ξ-th element of x. Since θ is regular, there is some ξ such that $\{x(\xi) : x \in \mathscr{A}_1\}$ is unbounded in θ. Now fix ξ_0 to be the least such ξ (ξ_0 may be 0). Let

$$\alpha_0 = \sup\{x(\eta) + 1 : x \in \mathscr{A}_1 \wedge \eta < \xi_0\};$$

then $\alpha_0 < \theta$ and $x(\eta) < \alpha_0$ for all $x \in \mathscr{A}_1$ and all $\eta < \xi_0$.

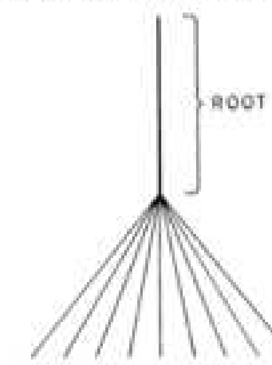

Figure 1.1. A Δ-System.

EXAMPLE SHEET #3.

(38) If κ is a cardinal, we say that $\mathbb{P}$ has the κ-c.c. if every antichain in $\mathbb{P}$ has cardinality smaller than κ. (Thus, the c.c.c. is the $\aleph_1$-c.c.) If κ is a cardinal in M, we say that $\mathbb{P}$ *preserves cardinals* $\geq \kappa$ if for every $\mathbb{P}$-generic filter G over M and every $\lambda \geq \kappa$, we have that $M \models$ "λ is a cardinal" if and only if $M[G] \models$ "λ is a cardinal". Show that if $M \models$ "κ is a regular cardinal" and $M \models$ "$\mathbb{P}$ has the κ-c.c.", then $\mathbb{P}$ preserves cardinals $\geq \kappa$.

This ΔSL gives with same proof as before:
If $M \models 2^{\aleph_1} = \aleph_2$, then $Fn(\aleph_1 \times \aleph_3, 2, \aleph_1)$ has the $\aleph_2^M$-c.c.

6.10. LEMMA. $\mathrm{Fn}(I, J, \lambda)$ *has the* $(|J|^{<\lambda})^+$-*c.c.*

PROOF. Let $\theta = (|J|^{<\lambda})^+$, and suppose that $\{p_\xi : \xi < \theta\}$ formed an antichain. First, assume λ is regular. Then $(|J|^{<\lambda})^{<\lambda} = |J|^{<\lambda}$, so $\forall \alpha < \theta (|\alpha^{<\lambda}| < \theta)$, so by the Δ-system lemma (see II 1.6) there is an $X \subset \theta$ with $|X| = \theta$ such that $\{\mathrm{dom}(p_\xi) : \xi \in X\}$ forms a Δ-system with some root r. Since there are less than θ possibilities for $p_\xi \restriction r$, we have a contradiction as in the proof for $\lambda = \omega$ (see Lemma 5.4).

If λ is singular, then since θ is regular and $> \lambda$, we could find a regular $\lambda' < \lambda$ such that $Y = \{\xi : |p_\xi| < \lambda'\}$ has cardinality θ. Then $\{p_\xi : \xi \in Y\}$ contradicts the $(|J|^{<\lambda'})^+$-c.c. which we have just proved for regular λ'. $\square$

<u>So:</u> Forcing over a model of GCH [or at least $2^{\aleph_1} = \aleph_2$] $Fn(\aleph_1 \times \aleph_3, 2, \aleph_1)$ preserves cardinals $\geq \aleph_2$.

CLOSURE

Def. A forcing $\mathbb{P}$ is called <u>λ-closed</u> if any family $\{p_\alpha ; \alpha<\gamma\}$ for $\gamma<\lambda$ that is a descending chain:

$$\alpha<\beta \implies p_\beta < p_\alpha$$

there is q s.t. $q \leq p_\alpha$ $\forall \alpha<\gamma$.

Example $\mathrm{Fn}(\aleph_1\times\aleph_3, 2, \aleph_1)$ is $\aleph_1$-closed.

[If $\{p_\alpha\}$ is a descending chain, $\bigcup_{\alpha<\gamma} p_\alpha$ is a condition in $\mathrm{Fn}(\aleph_1\times\aleph_3,2,\aleph_1)$.]

Theorem If $\mathbb{P}$ is λ-closed and $\kappa<\lambda$, then $\wp(\kappa)\cap M = \wp(\kappa)\cap M[G]$.

Corollary Forcing with $\mathrm{Fn}(\aleph_1\times\aleph_3, 2, \aleph_1)$

(a) does not change $\wp(\omega)$

(b) therefore preserves $\aleph_1$

[see ES#1 and the relation between codes for ctble wellorders and preserving $\aleph_1$]

To be proved in Lecture XVI.

Summary

Forcing with $Fn(\aleph_1 \times \aleph_3, 2, \aleph_1)$ over a model of GCH gives $M[G]$ with:

1. the same cardinals [since $M \models GCH$]
   [cardinals $\geq \aleph_2$ preserved by $\aleph_2$-c.c.;
   $\aleph_1$ preserved by $\aleph_1$-closure]
2. $2^{\aleph_1} \geq \aleph_3$ [standard]
3. $2^{\aleph_0} = \aleph_1$ [By Corollary 1. to the closure theor.]
4. Calculate number of nice names:

$$\aleph_3^{\aleph_1 \cdot \aleph_1} = \aleph_3^{\aleph_1} = \aleph_3 \cdot 2^{\aleph_1}$$

$$= \aleph_3 \cdot \aleph_2 = \aleph_3.$$

$$\Longrightarrow 2^{\aleph_1} = \aleph_3.$$

PROOF OF CLOSURE THEOREM STILL TO COME !!!