XIV

Fourteenth Lecture
FORCING & THE CONTINUUM HYPOTHESIS

Tuesday 11 March 2025

Lecture XIII

Suppose M ctm of ZFC

$\mathbb{P} := \mathrm{Fn}(\omega \times \aleph_2^M, 2)$

G $\mathbb{P}$-generic$/M$.

Then $M[G] \models 2^{\aleph_0} \geq \aleph_2$.

Remark In general, we can't have $M[G] \models 2^{\aleph_0} = \aleph_2$ here.

1. There is nothing special about $\aleph_2$:

   if $\mathbb{P} := \mathrm{Fn}(\omega \times \aleph_\alpha^M, 2)$, then

   $M[G] \models 2^{\aleph_0} \geq \aleph_\alpha$.

2. If $M \models 2^{\aleph_0} > \aleph_2$; which is possible by 1.,

   then $\mathcal{P}(\omega) \cap M \subseteq \mathcal{P}(\omega) \cap M[G]$.

   Therefore if $f \in M$ is s.t. $f : \aleph_3^M \longrightarrow \mathcal{P}(\omega) \cap M$ injection,

   then $f \in M[G]$ and f witnesses

   $M[G] \models 2^{\aleph_0} \geq \aleph_3$.

Q What are the possible values for $2^{\aleph_0}$?

Mentioned in L $\overline{\text{XIII}}$: not all values are possible

10. Explain why, for each $n \in \omega$, there is no surjection from $\aleph_n$ to $\aleph_{n+1}$. Use this fact to show that there is no surjection from $\aleph_\omega$ to $\aleph_\omega^{\aleph_0}$, and deduce that $2^{\aleph_0} \neq \aleph_\omega$.

$\aleph_\omega^{\aleph_0} > \aleph_\omega$

Note that $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$.

Gyula König

Born	16 December 1849 Győr, Kingdom of Hungary
Died	8 April 1913 (aged 63) Budapest, Austria-Hungary
Nationality	Hungarian
Alma mater	University of Heidelberg
Known for	König's paradox König's theorem (set theory) König's theorem (complex analysis)
Children	Dénes König

__Def.__ $C \subseteq \kappa$ is cofinal (= unbounded) if $\forall \lambda < \kappa \, \exists \gamma \in C \; \gamma \geq \lambda$.

$\mathrm{cf}\,\kappa := \min\{|C| ; C \text{ is cofinal}\}$

__Examples__ $\mathrm{cf}\,\aleph_1 = \aleph_1$

$\mathrm{cf}\,\aleph_\omega = \aleph_0$

Then LST ES#4 (10) is the special case $\kappa = \aleph_\omega$, $\mathrm{cf}\,\kappa = \aleph_0$ of König's lemma.

__König's Lemma__

$\kappa^{\mathrm{cf}\,\kappa} > \kappa$

Consequence for $2^{\mathrm{cf}\,\kappa}$:

$(2^{\mathrm{cf}\,\kappa})^{\mathrm{cf}\,\kappa} = 2^{\mathrm{cf}\,\kappa}$,

thus $2^{\mathrm{cf}\,\kappa} \neq \kappa$.

__Preview of A to Q.__

Every value not prohibited by König's Lemma is possible.

Now : $\aleph_2$!

Def. If $\mathbb{P}$ is any forcing, let
$$\mathcal{A} := \{A_n ; n \in \omega\}$$
be any ω-sequence of antichains in $\mathbb{P}$.
Let $\tau_{\mathcal{A}} := \{(\check{n}, p) ; p \in A_n\}$.
We call these nice names.

If $|\mathbb{P}| = \kappa$ and $\mathbb{P}$ has c.c.c., then there are at most $\kappa^{\aleph_0}$ many antichains and thus at most $(\kappa^{\aleph_0})^{\aleph_0} = \kappa^{\aleph_0 \cdot \aleph_0} = \kappa^{\aleph_0}$ many ω-seq. of antichains and thus nice names.

Theorem If $M[G] \models x \subseteq \omega$, then there is a nice name τ s.t. $\mathrm{val}(\tau, G) = x$.

Note: The Theorem does not need any assumptions about $\mathbb{P}$.

Proof. Start with μ s.t. $\mathrm{val}(\mu, G) = x$
possibly not nice

Fix $n \in \omega$. Use either a well-ordering of $\mathbb{P}$ or AC (to get one) and build a maximal antichain A_n s.t. $\forall p \in A_n \; p \Vdash \check{n} \in \mu$.

By DT, $\mathcal{A} := \{A_n ; n \in \omega\}$ is in M, so $\tau_{\mathcal{A}}$ is a $\mathbb{P}$-name in M. Of course: nice!

<u>Claim</u> $\mathrm{val}(\mu, G) = \mathrm{val}(\tau_{\mathfrak{A}}, G)$.

$\tau_{\mathfrak{A}} = \{(\check{u}, p) ; p \in A_u\}$

"$\supseteq$". If $u \in \mathrm{val}(\tau_{\mathfrak{A}}, G)$, then there is $p \in G$ s.t.

$(\check{u}, p) \in \tau_{\mathfrak{A}} \implies p \in A_u$
$\implies p \Vdash \check{u} \in \mu$

$u \in \mathrm{val}(\mu, G)$.

"$\subseteq$". If $u \in \mathrm{val}(\mu, G)$. So, by FT, get $q \in G$ s.t. $q \Vdash \check{u} \in \mu$.

<u>Subclaim</u> $A_u \cap G \neq \emptyset$

[By our lemma on incompatibility (ES#3), get $q' \in G$ s.t. $q' \perp p$ for all $p \in A_u$. Find $r \leq q, q'$, then $r \Vdash \check{u} \in \mu$. But that's a contradiction to A_u being maximal.]

So find $p \in G \cap A_u$. By def., $(\check{u}, p) \in \tau_{\mathfrak{A}}$
$+ p \in G$
$\rightarrow u \in \mathrm{val}(\tau_{\mathfrak{A}}, G)$

q.e.d.

Corollary If $\mathbb{P}$ has c.c.c. and

$$M \models |\mathbb{P}| = \kappa \wedge \lambda = \kappa^{\aleph_0},$$

then $M[G] \models 2^{\aleph_0} \leq \lambda$.

Pf. Follows directly from

(a) Theorem

(b) calculation of the number of nice names. q.e.d.

MAIN APPLICATION

If $\mathbb{P} = \mathrm{Fn}(\omega \times \aleph_2^M, 2)$, then

$$|\mathbb{P}| = \aleph_2^M.$$

Calculate in M, $\aleph_2^{\aleph_0}$.

By this calculation,

if $M \models 2^{\aleph_0} \leq \aleph_2$,

then $M[G] \models 2^{\aleph_0} \leq \aleph_2$.

Hausdorff's Formula

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_\beta}$$

$$\aleph_1^{\aleph_0} = \aleph_1 \cdot \aleph_0^{\aleph_0} = 2^{\aleph_0}$$

$$\aleph_2^{\aleph_0} = \aleph_2 \cdot \aleph_1^{\aleph_0} = \aleph_2 \cdot 2^{\aleph_0}$$

So $\aleph_2^{\aleph_0} = \max(\aleph_2, 2^{\aleph_0})$.

Corollary If $M \models CH$,

then $M[G] \models$

$$2^{\aleph_0} = \aleph_2.$$

<u>Remark 1</u> This proof also shows that if

$$M \models 2^{\aleph_0} \leq \aleph_n$$

and G is $\mathbb{P}$-gen./M where

$\mathbb{P} = \mathrm{Fn}(\omega \times \aleph_n^M, 2)$, then

$$M[G] \models 2^{\aleph_0} \leq \aleph_n.$$

<u>Cor</u> If $M \models CH$, then $M[G] \models 2^{\aleph_0} = \aleph_n$.

<u>Remark 2</u> What happens at $\aleph_\omega$?

$\mathbb{P} := \mathrm{Fn}(\omega \times \aleph_\omega^M, 2)$ $\quad |\mathbb{P}| = \aleph_\omega^M$

By general theory $M[G] \models 2^{\aleph_0} \geq \aleph_\omega$,

but König's lemma gives $2^{\aleph_0} \geq \aleph_{\omega+1}$.

What about the lower bound?

Our theorem & counting of nice names yields

$$M[G] \models 2^{\aleph_0} \leq \underbrace{\aleph_\omega^{\aleph_0}}_{> \aleph_\omega}$$

If $M \models GCH$, then

$$\aleph_\omega^{\aleph_0} \leq \aleph_\omega^{\aleph_\omega} = \aleph_{\omega+1}$$

So by König, $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}$.

Therefore, if $M \models GCH$, then $M[G] \models 2^{\aleph_0} = \aleph_{\omega+1}$.

First limit cardinal that is a possible value of $2^{\aleph_0}$ is $\aleph_{\omega_1}$

Clearly, $\mathbb{P} = \mathrm{Fn}(\omega \times \aleph_{\omega_1}^M, 2)$
adds inj. from $\aleph_{\omega_1}^M$ into $\mathcal{P}(\omega)$,
so $M[G] \models 2^{\aleph_0} \geq \aleph_{\omega_1}$.

Count nice names : $|\mathbb{P}|^{\aleph_0} = \aleph_{\omega_1}^{\aleph_0}$.

If for all $\alpha < \omega_1$, $\aleph_\alpha^{\aleph_0} \underset{(*)}{\leq} \aleph_{\omega_1}$, then
$\aleph_{\omega_1}^{\aleph_0} = \aleph_{\omega_1}$.

$$\aleph_{\omega_1}^{\aleph_0} = X = \{f ; f : \omega \longrightarrow \aleph_{\omega_1}\} = \bigcup_{\alpha < \omega_1} \{f ; f : \omega \longrightarrow \aleph_\alpha\}$$

[since ω_1 has no cofinal ω-seq.]

Thus $|X| \leq \aleph_1 \cdot \aleph_{\omega_1} = \aleph_{\omega_1}$.
[by assumption (*)]

Thus : $M[G] \models 2^{\aleph_0} = \aleph_{\omega_1}$.