XIII

Thirteenth Lecture
Forcing & the Continuum Hypothesis
Saturday 8 March 2025

<u>Recap</u>

Lecture <u>XII</u>

RECAP

FT FORCING THEOREM
$M[G]\models\varphi \iff \exists p\in G\ p\Vdash\varphi$

SynFT SYNTACTIC FORCING THEOREM
$M[G]\models\varphi \iff \exists p\in G\ M\models p\Vdash^{*}\varphi$

DT DEFINABILITY THEOREM
$p\Vdash\varphi \iff p\Vdash^{*}\varphi$

GMT GENERIC MODEL THEOREM
$M[G]\models \mathrm{ZFC}$

Lecture <u>VIII</u>

EXAMPLE 3.

$\mathrm{Fn}(X\times Y, 2)$

Assume X is infinite.
Consider $E_{y,y'} := \{p\in\mathbb{P};\ \exists x\in X\ p(x,y)\neq p(x,y')\}$

This is dense for $y\neq y'$.

$\mathcal{E} := \{E_{y,y'};\ y\neq y'\in Y\}$

<u>Lemma 5</u> If F is $\mathcal{D}\cup\mathcal{E}$-generic, then there is an injection from Y into $\wp(X)$.

<u>Summary</u> If G is $\mathrm{Fn}(\omega\times\aleph_2^M, 2)$-generic, then

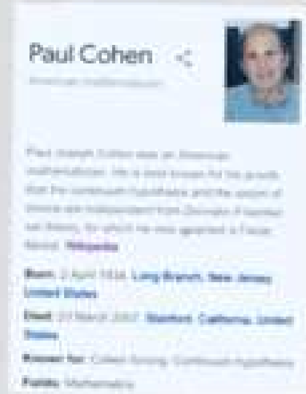
$M[G]\models \mathrm{ZFC}$ + there is an inj. from $\aleph_2^M$ into $\wp(\omega)$.
[GMT]

This implies $M[G]\models \mathrm{ZFC} + 2^{\aleph_0}\geq\aleph_2$
if we have $\aleph_1^M = \aleph_1^{M[G]}$
$\aleph_2^M = \aleph_2^{M[G]}$.

Goal for L <u>XIII</u>

REMARK on our metamathematical application.

Lecture VIII :



Cohen

$\forall T \subseteq ZFC$ finite $\exists T^* \subseteq ZFC$ finite s.t.

(*) if M ctm of T^*, then there is $N \supseteq M$ ctm of $T + \neg CH$

We have seen (ES#1) that (*) implies $Con(ZFC) \implies Con(ZFC + \neg CH)$.

SIMPLIFIED: If M ctm of ZFC, then there is $N \supseteq M$ ctm of $ZFC + \neg CH$

Note that our proof from page 1 is NOT GOOD ENOUGH for the statement (*) from lecture VIII.

For this, we need to look more carefully at the proof of the GMT:

$$M \models ZFC \implies M[G] \models ZFC.$$

The proof proceeds

AXIOM BY AXIOM

and thus for each $\varphi \in ZFC$, we find finite S_φ s.t. $M \models S_\varphi \implies M[G] \models \varphi$.

Let $S \subseteq ZFC$ finite s.t. S proves that all relevant notions (name, value, ...) are well-defined and absolute. Then for $T \subseteq ZFC$ finite, define

$$T^* := S \cup \bigcup_{\varphi \in T} S_\varphi .$$

Then, the proof shows (*).

Let's prove $\neg$ CH in $M[G]$.

Def. We say $\mathbb{P}$ preserves cardinals if $\forall G$ $\mathbb{P}$-generic over M, "κ is a cardinal" is absolute betw. M & $M[G]$.

Def. We say $\mathbb{P}$ has the countable chain condition (c.c.c.) if every antichain in $\mathbb{P}$ is countable.

Theorem 1 If $\mathbb{P}$ has c.c.c., then $\mathbb{P}$ preserves cardinals.

Theorem 2 $Fn(\omega \times \aleph_2^M, 2)$ has the c.c.c.

Corollary If G is $Fn(\omega \times \aleph_2^M, 2)$-generic over M, then
$M[G] \models ZFC + 2^{\aleph_0} \geq \aleph_2$.

Lemma (for Thm 1) $M \models \mathbb{P}$ has c.c.c., $X, Y \in M$, G $\mathbb{P}$-generic over M $f: X \longrightarrow Y$, $f \in M[G]$.
Then there is $F \in M$ s.t. $\forall x \in X$ $F(x) \subseteq Y$
$\forall x \in X$ $f(x) \in F(x)$ (*)
$M \models \forall x \in X$ $F(x)$ is countable.

Proof of Thm 1 from Lemma

Suppose $M \models \kappa$ is a cardinal, $M[G] \models \kappa$ is not a cardinal, so there is $\lambda < \kappa$ and $f \in M[G]$, $f: \lambda \to \kappa$, f is a surjection. Apply Lemma and get F. Define $R := \bigcup_{\alpha < \lambda} F(\alpha)$. By (*), $R = \kappa$.
But $M \models |R| = \aleph_0 \cdot \lambda = \lambda < \kappa$. But then $M \models \kappa$ is not a cardinal.
Contradiction. q.e.d.

Lemma $M \models \mathbb{P}$ has c.c.c, $X, Y \in M$
G $\mathbb{P}$-gen. $/M$, $f: X \to Y$, $f \in M[G]$
Then there is $F \in M$ s.t.
$\forall x \in X$ $f(x) \in F(x)$ (1)
$F(x) \subseteq Y$ (2)
$M \models F(x)$ is countable. (3)

Proof. Let $F(x) := \{ y \in Y ; \exists p \in \mathbb{P} \; p \Vdash \tau(\check{x}) = \check{y} \}$
Fix τ, a name for f. By DT, $F \in M$.

(2) follows from definition.
(1) follows from FT.
Let's look at (3): Since $M[G] \models f$ is a function,
find $q \Vdash \tau$ is a function. with $q \in G$.
For each $y \in F(x)$, pick $p_y \leq q$ s.t.
$$p_y \Vdash \tau(\check{x}) = \check{y}.$$
Note that if $y \neq y'$, then $p_y \perp p_{y'}$.
Thus $\{ p_y ; y \in F(x) \}$ is an antichain, thus countable in M. So $F(x)$ is countable in M.

q.e.d.

Theorem 2 $Fn(\omega \times \aleph_2^M, 2)$ has c.c.c.

Much more: If Y is countable, then $Fn(X, Y)$ has c.c.c.

Proof. Δ-systems from ~~Example~~ (33) are also called quasi-disjoint families

ΔSL Any uncountable family of finite sets contains an uncountable Δ-system.

Example Sheet #3:

(33) A family of finite sets $\mathcal{D}$ is called a *Δ-system* if there is a finite set R (called the *root of the Δ-system*) such that for all $D, D' \in \mathcal{D}$, if $D \neq D'$, then $D \cap D' = R$. Show that any uncountable family of finite sets contains an uncountable Δ-system.

(*Hint.* Argue that you can assume w.l.o.g. that all elements of the family have the same size and prove the claim by induction on the size of the elements of the family.)

Take any $A \subseteq \mathbb{P}$ uncountable and prove that it's not an antichain.

If $p \in A$, then $\mathrm{dom}(p) \subseteq X$ finite.

Consider $S := \{\mathrm{dom}(p) ; p \in A\}$. That's an uncountable family of finite sets, so by ΔSL find ΔS $\mathcal{D} \subseteq S$ uncountable.

Let $r \subseteq X$ finite be the root of $\mathcal{D}$.

Since Y is countable, there are only countably many functions $q : r \longrightarrow Y$. Since $\mathcal{D}$ is uncountable, by PHP, there are p, q s.t. $\mathrm{dom}(p), \mathrm{dom}(q) \in \mathcal{D}$ and $p \restriction r = q \restriction r$. But since $\mathrm{dom}(p) \cap \mathrm{dom}(q) = r$, p and q are compatible. So A is not an antich.

q.e.d.

Next time

We got $M[G] \models 2^{\aleph_0} \geq \aleph_2$.

What is the size of $2^{\aleph_0}$ in $M[G]$?

Remember (L&ST ES#4.
$ZFC \vdash 2^{\aleph_0} \neq \aleph_\omega$.)

What if $\aleph_2$ was one of the forbidden values?

Note Obtaining $M[G] \models 2^{\aleph_0} = \aleph_2$ cannot be quite as general as this proof: if $M \models 2^{\aleph_0} > \aleph_2$, then this will remain true in $M[G]$.

Note 2 If G is $Fn(\omega \times \aleph_\alpha^M, 2)$-generic over M, then
$$M[G] \models 2^{\aleph_0} \geq \aleph_\alpha.$$