FCH $\overline{\text{XII}}$

Twelfth lecture of Forcing & CH
4 March 2025

RECAP

FT FORCING THEOREM

$$M[G] \models \varphi \iff \exists p \in G \; p \Vdash \varphi$$

SynFT SYNTACTIC FORCING THEOREM

$$M[G] \models \varphi \iff \exists p \in G \; M \models p \Vdash^* \varphi$$

DT DEFINABILITY THEOREM

$$p \Vdash \varphi \iff p \Vdash^* \varphi$$

GMT GENERIC MODEL THEOREM

$$M[G] \models ZFC$$

Already done

$$FT \implies GMT$$

$$SynFT \implies DT$$

$$SynFT + DT \implies FT$$

So, remains to show: SynFT

Equality

$p \Vdash^* \tau_0 = \tau_1 \iff \forall (\pi_0, s_0) \in \tau_0$
$\{q \leq p ;\, q \leq s_0 \longrightarrow \exists (\pi_1, s_1) \in \tau_1 \, (q \leq s_1 \wedge q \Vdash^* \pi_0 = \pi_1)\}$
is dense below p and $\forall (\pi_1, s_1) \in \tau_1$
$\{q \leq p ;\, q \leq s_1 \longrightarrow \exists (\pi_0, s_0) \in \tau_0 \, (q \leq s_0 \wedge q \Vdash^* \pi_0 = \pi_1)\}$
is dense below p.

Element

$p \Vdash^* \tau_0 \in \tau_1 :\iff$
$\{q ;\, \exists (\pi, s) \in \tau_1 \, (q \leq s \wedge q \Vdash^* \pi = \tau_0)\}$
is dense below p.

$p \Vdash^* \varphi(\vec{\tau}) \wedge \psi(\vec{\tau}) :\iff p \Vdash^* \varphi(\vec{\tau})$ and $p \Vdash^* \psi(\vec{\tau})$

$p \Vdash^* \neg \varphi(\vec{\tau}) :\iff \forall q \leq p \;\; q \not\Vdash^* \varphi(\vec{\tau})$

$p \Vdash^* \exists x \varphi(x, \vec{\tau}) :\iff \{r ;\, \exists \sigma \; r \Vdash^* \varphi(\sigma, \vec{\tau})\}$
is dense below p

Proof of SynFT:

$=$ Proof by induction on name rank

$\in$ Proof from $=$

$\neg, \wedge, \exists$ Proof by induction on formula complexity

$\neg$ done ✓

$\wedge, \exists$ ES#3

Syntactic Forcing Theorem

$$M[G] \models \varphi \iff \exists p \in G \quad M \models p \Vdash^* \varphi$$

Element

$p \Vdash^* \tau_0 \in \tau_1 \quad :\iff$

$D = \{q \,;\, \exists (\pi, s) \in \tau_1 \,(q \leq s \wedge q \Vdash^* \pi = \tau_0)\}$

is dense below p.

Assume SynFT for "=" and prove it for "$\in$".

"$\Longrightarrow$". Assume that $M[G] \models \tau_0 \in \tau_1$.

i.e. $\mathrm{val}(\tau_0, G) \in \mathrm{val}(\tau_1, G)$.

Thus, there is $(\pi, s) \in \tau_1$ s.t. $\mathrm{val}(\pi, G) = \mathrm{val}(\tau_0, G)$ and $s \in G$.

$\Updownarrow$

$M[G] \models \pi = \tau_0$

So, by SynFT for $=$, find $r \in G$ s.t.

$r \Vdash^* \pi = \tau_0$.

Find $p \leq r, s$ s.t. $p \in G$.

Claim that D is dense below p. Let $q \in D$ and pick the above (π, s). Then we have $q \leq p \leq s$ and $q \Vdash^* \pi = \tau_0$ (since $q \leq p \leq r$).

"$\Longleftarrow$". Assume $p \in G \quad p \Vdash^* \tau_0 \in \tau_1$, i.e.,

D is dense below p.

By Lemma (iv), find $q \in G \cap D$, thus there is $(\pi, s) \in \tau_1$ s.t.

$q \leq s$, $q \Vdash^* \pi = \tau_0$.

Since $q \in G$, $s \in G$, so

$\mathrm{val}(\pi, G) \in \mathrm{val}(\tau_1, G)$

By SynFT for "=", have

$\mathrm{val}(\pi, G) = \mathrm{val}(\tau_0, G)$.

$\Rightarrow \mathrm{val}(\tau_0, G) \in \mathrm{val}(\tau_1, G)$. q.e.d.

LEMMA G $\mathbb{P}$-generic$/M$; $E \subseteq \mathbb{P}$, $E \in M$

(i) If E is dense below p and $q \leq p$, then E is dense below q.

(ii) If $\{r \,;\, E \text{ is dense below } r\}$ is dense below p, then E is dense below p.

(iii) Either $G \cap E \neq \emptyset$ or there is $q \in G$ s.t. $\forall r \in E \;\; r \perp q$.

(iv) If $p \in G$ and E is dense below p, then $G \cap E \neq \emptyset$.

Equality

SynFT: $M[G] \models \tau_0 = \tau_1 \iff \exists p \in G$
$p \Vdash^* \tau_0 = \tau_1$.

$p \Vdash^* \tau_0 = \tau_1 \;:\!\iff\; \forall (\pi_0, s_0) \in \tau_0$

$D_0 = \{ q \leq p \,;\, q \leq s_0 \longrightarrow \exists (\pi_1, s_1) \in \tau_1 \, (q \leq s_1 \wedge q \Vdash^* \pi_0 = \pi_1) \}$

is dense below p and $\forall (\pi_1, s_1) \in \tau_1$

$D_1 = \{ q \leq p \,;\, q \leq s_1 \longrightarrow \exists (\pi_0, s_0) \in \tau_0 \, (q \leq s_0 \wedge q \Vdash^* \pi_0 = \pi_1) \}$

is dense below p.

We prove this by induction on name rank, so assume that SynFT for "=" is true for names π_0, π_1 with smaller name rank.

"$\Leftarrow$". Assume $p \in G$ s.t. $p \Vdash^* \tau_0 = \tau_1$. To prove $\mathrm{val}(\tau_0, G) = \mathrm{val}(\tau_1, G)$.

We'll show that the fact that D_0 is dense below p implies

CLAIM $\mathrm{val}(\tau_0, G) \subseteq \mathrm{val}(\tau_1, G)$.

The other direction is the same proof with 0/1 flipped.

Proof of Claim. Let $x \in \mathrm{val}(\tau_0, G)$, so find $(\pi_0, s_0) \in \tau_0$ s.t. $s_0 \in G$ and $x = \mathrm{val}(\pi_0, G)$. So, the corresponding D_0 is dense below p.

Find $q \leq s_0, p$ s.t. $q \in G$. Then D_0 is dense below q.

So find $r \leq q$ s.t. $r \in G \cap D_0$. [Note that $r \leq q \leq s_0 \Rightarrow r \leq s_0$]

Since $r \in D_0$ & $r \leq s_0$, find $(\pi_1, s_1) \in \tau_1$ s.t.

$$r \leq s_1 \wedge r \Vdash^* \pi_0 = \pi_1.$$

By IH, $r \in G$ & $r \Vdash^* \pi_0 = \pi_1$ implies

$$x = \mathrm{val}(\pi_0, G) = \mathrm{val}(\pi_1, G) \in \mathrm{val}(\tau_1, G)$$

[since $(\pi_1, s_1) \in \tau_1$ & $s_1 \in G$]

Equality

$p \Vdash^* \tau_0 = \tau_1 :\Longleftrightarrow \forall (\pi_0, s_0) \in \tau_0$

$D_0 = \{ q \leq p ; q \leq s_0 \longrightarrow \exists (\pi_1, s_1) \in \tau_1 \; (q \leq s_1 \wedge q \Vdash^* \pi_0 = \pi_1) \}$

is dense below p and $\forall (\pi_1, s_1) \in \tau_1$

$D_1 = \{ q \leq p ; q \leq s_1 \longrightarrow \exists (\pi_0, s_0) \in \tau_0 \; (q \leq s_0 \wedge q \Vdash^* \pi_0 = \pi_1) \}$

is dense below p.

SynFT $\mathrm{val}(\tau_0, G) = \mathrm{val}(\tau_1, G)$

$\Longleftrightarrow$

$\exists p \in G \; p \Vdash^* \tau_0 = \tau_1$

"$\Longrightarrow$". Assume $\mathrm{val}(\tau_0, G) = \mathrm{val}(\tau_1, G)$.

Consider the set

$$D := \{ r ; \; r \Vdash^* \tau_0 = \tau_1 \quad \text{OR}$$

$$\Phi_r^0 \quad \exists (\pi_0, s_0) \in \tau_0 \, (r \leq s_0 \wedge \forall (\pi_1, s_1) \in \tau_1 \; \forall q \, ((q \leq s_1 \wedge q \Vdash^* \pi_0 = \pi_1) \rightarrow q \perp r)$$

$$\text{OR}$$

$$\Phi_r^1 \quad \exists (\pi_1, s_1) \in \tau_1 \, (r \leq s_1 \wedge \forall (\pi_0, s_0) \in \tau_0 \; \forall q \, ((q \leq s_0 \wedge q \Vdash^* \pi_0 = \pi_1) \rightarrow q \perp r) \}$$

D is dense. If $p \Vdash^* \tau_0 = \tau_1$, then $p \in D$, so nothing to show. If $p \not\Vdash^* \tau_0 = \tau_1$, then there is some D_0 / D_1 that fails to be dense below p.

CLAIM 1 We'll show: If some D_0 is not dense below p, find $r \leq p$ s.t. Φ_r^0 holds.

[Other proof: "some D_1 is not dense below p $\rightsquigarrow \exists r \leq p$ s.t. Φ_r^1 holds" is again flipping 0/1 in the proof.]

Equality

$p \Vdash^* \tau_0 = \tau_1 :\Longleftrightarrow \forall (\pi_0, s_0) \in \tau_0$

$\mathcal{D}_0 = \{ q \leq p ; q \leq s_0 \longrightarrow \exists (\pi_1, s_1) \in \tau_1 \; (q \leq s_1 \wedge q \Vdash^* \pi_0 = \pi_1) \}$

is dense below p and $\forall (\pi_1, s_1) \in \tau_1$

$\{ q \leq p ; q \leq s_1 \longrightarrow \exists (\pi_0, s_0) \in \tau_0 \; (q \leq s_0 \wedge q \Vdash^* \pi_0 = \pi_1) \}$

is dense below p.

We're showing that

some $\mathcal{D}_0$ not dense below p

$\Longrightarrow$

$\exists r \leq p$ s.t. Φ_r^0

$\Phi_r^0 \quad \exists (\pi_0, s_0) \in \tau_0 \left[r \leq s_0 \wedge \forall (\pi_1, s_1) \in \tau_1 \; \forall q \; ((q \leq s_1 \wedge q \Vdash^* \pi_0 = \pi_1 \longrightarrow q \perp r) \right]$

Suppose $p \not\Vdash^* \tau_0 = \tau_1$ and there is $(\pi_0, s_0) \in \tau_0$ s.t. $\mathcal{D}_0$ is not dense below p.

This means that there is some $r \leq p$ s.t.

(*) $\forall q \leq r \; (q \leq s_0 \wedge \forall (\pi_1, s_1) \in \tau_1 \; \neg (q \leq s_1 \wedge q \Vdash^*_{\pi_0 = \pi_1}))$

$\rightarrow r \leq s_0$ [Since it holds for all $q \leq r$.]

Fix any $(\pi_1, s_1) \in \tau_1$ and any q s.t. $q \leq s_1$ and $q \Vdash^* \pi_0 = \pi_1$. (**)

If there was a common extension of q and r, say q_0, then by (*) $q_0 \not\Vdash^* \pi_0 = \pi_1$ and by (**), $q_0 \Vdash^* \pi_0 = \pi_1$. Contradiction

Thus $q \perp r$.

[THIS FINISHES THE PROOF OF CLAIM 1 : $\mathcal{D}$ IS DENSE.]

Equality

$p \Vdash^* \tau_0 = \tau_1 :\Longleftrightarrow \forall (\pi_0, s_0) \in \tau_0$
$\{q \le p \,;\, q \le s_0 \longrightarrow \exists (\pi_1, s_1) \in \tau_1 \ (q \le s_1 \wedge q \Vdash^* \pi_0 = \pi_1)\}$
is dense below p and $\forall (\pi_1, s_1) \in \tau_1$
$\{q \le p \,;\, q \le s_1 \longrightarrow \exists (\pi_0, s_0) \in \tau_0 \ (q \le s_0 \wedge q \Vdash^* \pi_0 = \pi_1)\}$
is dense below p.

$\Phi^0_r := \exists (\pi_0, s_0) \in \tau_0 \ (r \le s_0 \wedge \forall (\pi_1, s_1) \in \tau_1 \forall q ((q \le s_1 \wedge q \Vdash^* \pi_0 = \pi_1) \longrightarrow q \perp r)$

Summary We now have that D is dense.
$D = \{r \,;\, r \Vdash^* \tau_0 = \tau_1 \text{ or } \Phi^0_r \text{ or } \Phi^1_r\}$

Claim 2 If $r \in G$, then neither Φ^0_r nor Φ^1_r holds.

Proof of Claim 2 [Again, we only show $\neg \Phi^0_r$; the other proof is flipping 0/1.]

If Φ^0_r holds, then find $(\pi_0, s_0) \in \tau_0$ s.t. $r \le s_0$
(*) and $\forall (\pi_1, s_1) \in \tau_1 \forall q ((q \le s_1 \wedge q \Vdash^* \pi_0 = \pi_1) \longrightarrow q \perp r)$

Since we assumed $r \in G$, we obtain $s_0 \in G$ (since $r \le s_0$), so $\mathrm{val}(\pi_0, G) \in \mathrm{val}(\tau_0, G)$.

We use our assumption $\mathrm{val}(\tau_0, G) = \mathrm{val}(\tau_1, G)$ to obtain $(\pi_1, s_1) \in \tau_1$ with $s_1 \in G$ s.t. $\mathrm{val}(\pi_0, G) = \mathrm{val}(\pi_1, G)$.

Applying IH to $\mathrm{val}(\pi_0, G) = \mathrm{val}(\pi_1, G)$, we get $q_0 \in G$ s.t. $q_0 \Vdash^* \pi_0 = \pi_1$.

Now find $q \le q_0, s_1$, $q \in G$. Since $q \le q_0$, we have $q \Vdash^* \pi_0 = \pi_1$.

$q \le s_1$ + (*) $\Longrightarrow q \perp r$.

But both $q, r \in G$. Contradiction! [Finishes the proof of Claim 2.]

Put everything together:

Since D is dense by Claim 1, find
$$r \in D \cap G.$$
Therefore, by Claim 2, Φ_r^0, Φ_r^1 do not hold.

Thus $r \Vdash^* \tau_0 = \tau_1$.

q.e.d.
(Syntactic Forcing Theorem)

Remark. So, now we know that if
$$\mathbb{P} = \mathrm{Fn}(\aleph_2^M \times 2, \aleph_0)$$
and G is $\mathbb{P}$-generic over M, then
$$M[G] \models \mathrm{ZFC} + \text{there is an injection from } \aleph_2^M \text{ into } \mathcal{P}(\omega).$$

This is only CH, if $\aleph_1^M = \aleph_1^{M[G]}$ and $\aleph_2^M = \aleph_2^{M[G]}$.

The following lectures will establish that.