XI

ELEVENTH LECTURE
FORCING & THE CONTINUUM HYPOTHESIS

1 March 2025

Lecture X, page 5

FORCING Fix M ctm of ZFC, $\mathbb{P} \in M$ forcing poset.
We call the language
$$\mathcal{L}^M_{\text{Forcing}} := \mathcal{L}_\in \cup \{\tau \,;\, \mathbb{P}\text{-name}\}$$
the FORCING LANGUAGE (OVER M).

Interpretation: If G is a $\mathbb{P}$-generic over M and φ is in the forcing language, we say
$$M[G] \models \varphi$$
$$\iff M[G], v \models \varphi$$
where $v(\tau) := \mathrm{val}(\tau, G)$.

SEMANTIC FORCING PREDICATE:
$$p \Vdash_{M,\mathbb{P}} \varphi \;:\iff\; \forall G\ \mathbb{P}\text{-generic}/M \text{ with } p \in G\quad M[G] \models \varphi.$$
"p forces φ"
[We often omit $M, \mathbb{P}$.]

Two theorems at the heart of forcing:

1. Forcing Theorem: If G is $\mathbb{P}$-generic over M, then $M[G] \models \varphi \iff \exists p \in G\ p \Vdash \varphi$
2. Definability Theorem: "$p \Vdash_M \varphi$" is absolutely definable for ctms models containing M.

Plan Prove the FT & DT.
[Will do this via a technical auxiliary notion known as SYNTACTIC FORCING]

Currently: Proving $M[G] \models \mathrm{ZFC}$ from FT.

$M[G] \models \mathrm{ZFC}$

- Extensionality ✓ Done (transitivity)
- Foundation ✓ Done (transitivity)
- Pairing ✓ Done (easy)
- Union ✓ Done (slightly harder)
- Choice ✓ Done (easy)
- Separation DONE (L X)
- Replacement ??
- Powerset ??

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Proof of Separation in $M[G]$.
Let $\varphi(x, x_1, \dots, x_n)$ be an $\mathcal{L}_\in$-formula and $x \in M[G]$.
Need name for
$$A := \{z \in x \,;\, M[G] \models \varphi(z, \vec{x})\}$$
[For readability drop parameters]

Fix σ s.t. $\mathrm{val}(\sigma, G) = x$.
Define $\rho := \{(\tau, p) \,;\, \tau \in \mathrm{dom}(\sigma) \wedge p \Vdash \tau \in \sigma \wedge \varphi(\tau)\}$

By DT, ρ is a name in M.

Claim: $\mathrm{val}(\rho, G) = A$.

"$\subseteq$" If $z \in \mathrm{val}(\rho, G)$, then there is $(\tau, p) \in \rho$ s.t. $z = \mathrm{val}(\tau, G)$, $p \in G$.
$\tau \in \mathrm{dom}(\sigma)$, $p \Vdash \tau \in \sigma \wedge \varphi(\tau)$
$M[G] \models \tau \in \sigma \wedge \varphi(\tau) \implies M[G] \models z \in x \wedge \varphi(z) \implies z \in A$.

"$\supseteq$" If $z \in A \implies z \in x \implies$ there is $\tau \in \mathrm{dom}(\sigma)$, $z = \mathrm{val}(\tau, G)$
+ $M[G] \models \varphi(z)$
$\overset{\text{FT}}{\implies} \exists p \in G\ p \Vdash \varphi(\tau)$
Also there is $q \in G$ s.t. $q \Vdash \tau \in \sigma$.
Find $r \in G$, $r \leq p, q$ s.t. $r \Vdash \tau \in \sigma \wedge \varphi(\tau)$
Then $(\tau, r) \in \rho$ by def., so
$z = \mathrm{val}(\tau, G) \in \mathrm{val}(\rho, G)$. q.e.d.

Proof of Powerset

$\longrightarrow$ ES#3.

Proof of REPLACEMENT.

Since we already have Separation, it's enough to show the following:

if φ is functional formula and $x \in M[G]$, then there is $R \in M[G]$ s.t.

$$(*) \quad M[G] \models \forall y \in x \, \exists z \in R \, \varphi(y, z, \cancel{\vec{p}})$$

We work in M and identify a name ϱ for R. Fix σ s.t. $x = \mathrm{val}(\sigma, G)$. Find α large enough s.t. $\mathrm{dom}(\sigma) \subseteq V_\alpha$.

As before, suppress parameters for notational ease.

Consider

$$\psi(p, \pi) := \underbrace{\exists \mu \;\; p \Vdash \varphi(\pi, \mu)}_{\text{By DT, this is an } \mathcal{L}_\in\text{-formula.}}$$

$\in \mathbb{P}$ name

By LRT, find $\vartheta > \alpha$ s.t. ψ is absolute between V_ϑ and $V = M$.

Define $\varrho := \{(\mu, \mathbb{1}) ; \mu \in V_\vartheta\}$ and $R := \mathrm{val}(\varrho, G)$.

Proof of $(*)$: Fix $y \in x$, $y = \mathrm{val}(\pi, G)$. Since φ is functional, we know that $\varphi(\pi, \mu)$ holds in $M[G]$ for some μ. By FT, there is $p \in G$ s.t. $p \Vdash \varphi(\pi, \mu)$. So $M \models \psi(p, \pi)$. By absoluteness, $V_\vartheta \models \psi(p, \pi)$. This means $\exists \mu \in V_\vartheta$ s.t. $p \Vdash \varphi(\pi, \mu)$.

Thus: $\mathrm{val}(\mu, G) \in \mathrm{val}(\varrho, G) = R$. qed

Lecture X

We are going to do the following:

(a) Define the syntactic forcing predicate $\Vdash^*$ that is absolutely definable.

(b) Prove the FT for $\Vdash^*$

(c) Derive that $\Vdash \iff \Vdash^*$.

[Lectures XI & XII.]

Observation (under the assumption of FT):

① If $q \leq p$ and $p \Vdash \varphi$, then $q \Vdash \varphi$.

② If $p \Vdash \exists x \varphi(x)$, then there is a name τ and $q \leq p$ s.t. $q \Vdash \varphi(\tau)$.

[If $p \Vdash \exists x \varphi(x)$ and $p \in G$, then by def. $M[G] \models \exists x \varphi(x)$, so there is τ name s.t. $M[G] \models \varphi(\tau)$. By FT, find $r \in G$ s.t. $r \Vdash \varphi(\tau)$.
Find $q \leq p, r$, $q \in G$. By ① $q \Vdash \varphi(\tau)$.]

Def. $D \subseteq \mathbb{P}$ is called dense below p if $\forall q \leq p \; \exists r \leq q \; r \in D$.

Lemma If G is $\mathbb{P}$-gen./M, $E \subseteq \mathbb{P}$, $E \in M$. Then

(i) If E is dense below p, $q \leq p$, then E is dense below q.

(ii) If $\{r ; E \text{ is dense below } r\}$ is dense below p, then E is dense below p.

(iii) Either $G \cap E \neq \emptyset$ or $\exists q \in G \; \forall r \in E \; r \perp q$.

(iv) If $p \in G$, E is dense below p, then $G \cap E \neq \emptyset$.

$\hookrightarrow$ ES#3.

DEFINITION OF THE SYNTACTIC FORCING RELATION

$$p \Vdash^* \varphi(\vec{\tau})$$

Two recursions: first "$\Vdash^* =$" by recursion on Names.
then "$\Vdash^* \in$" without recursion,
then the rest by recursion on formula complexity.

$p \Vdash^* \tau_0 = \tau_1 \quad :\Longleftrightarrow$

$\forall (\pi_0, s_0) \in \tau_0$

$$\{ q \leq p \,;\, q \leq s_0 \longrightarrow \exists (\pi_1, s_1) \in \tau_1 \, (q \leq s_1 \wedge q \Vdash^* \pi_0 = \pi_1) \}$$

is dense below p

and

$\forall (\pi_1, s_1) \in \tau_1$

$$\{ q \leq p \,;\, q \leq s_1 \longrightarrow \exists (\pi_0, s_0) \in \tau_0 \, (q \leq s_0 \wedge q \Vdash^* \pi_0 = \pi_1) \}$$

is dense below p.

[Remark: This is a recursion on Names.]

$p \Vdash^* \tau_0 \in \tau_1 \quad :\Longleftrightarrow$

$$\{ q \leq p \,;\, \exists (\pi, s) \in \tau_1 \, (q \leq s \wedge q \Vdash^* \pi = \tau_0) \}$$

is dense below p.

[Remark: no recursion involved, just "$\Vdash^* =$".]

Recursion on complexity of formulas:

$$p \Vdash^* \varphi \wedge \psi \;:\Longleftrightarrow\; p \Vdash^* \varphi \text{ and } p \Vdash^* \psi$$

$$p \Vdash^* \neg\varphi \;:\Longleftrightarrow\; \forall q \leq p \;\; q \not\Vdash^* \varphi.$$

$$p \Vdash^* \exists x \varphi(x) \;:\Longleftrightarrow\; \{ r \,;\, \exists \sigma \;\; r \Vdash^* \varphi(\sigma) \}$$

is dense below p.

Remark: These definitions remind us of the definitions of Kripke semantics for intuitionistic logic.

<u>Lemma</u> TFAE

(i) $p \Vdash^* \varphi$

(ii) $\forall r \leq p \ r \Vdash^* \varphi$

(iii) $\{r ; r \Vdash^* \varphi\}$ is dense below p.

<u>Proof.</u> If this is true for $\varphi = \tau_0 = \tau_1$, $= \tau_0 \in \tau_1$, then it's true for all formulas.

For φ atomic, we get that

$(ii) \Longrightarrow (i)$
$(ii) \Longrightarrow (iii)$ is trivial.

$(i) \Longrightarrow (ii)$ follows from our Lemma on page 3, item (i).

$(iii) \Longrightarrow (i)$ follows from the same Lemma, item (ii).

q.e.d.

Strategy

1. **Syntactic Forcing Theorem**

$$M[G] \models \varphi \iff \exists p \in G \ M \models p \Vdash^* \varphi.$$

2. COROLLARY 1
[= Definability Theorem]

$$p \Vdash \varphi \iff p \Vdash^* \varphi.$$

$$[p \Vdash_M \varphi \iff M \models p \Vdash^* \varphi]$$

3. COROLLARY 2:
Forcing Theorem

$$M[G] \models \varphi \iff \exists p \in G \ p \Vdash \varphi.$$

Cor 2 is just Cor 1 + SynFT.

Proof of Cor 1 from SynFT:

$\Leftarrow$ is just SynFT.

$\Rightarrow$ Suppose $p \Vdash \varphi$. By Lemma, we need to show $\{r \leq p ; r \Vdash^* \varphi\}$ is dense below p.

Suppose not. Then I find $q \leq p$ s.t. $\forall r \leq q \ r \not\Vdash^* \varphi$. So $q \Vdash^* \neg\varphi$.

If $q \in G$, then $p \in G$

$M[G] \models \varphi$

$M[G] \models \neg\varphi$.

Contradiction!

q.e.d.

LEMMA G $\mathbb{P}$-generic/M ; $E \subseteq \mathbb{P}$, $E \in M$

(i) If E is dense below p and $q \leq p$, then E is dense below q.

(ii) If $\{r ; E$ is dense below $r\}$ is dense below p, then E is dense below p.

(iii) Either $G \cap E \neq \emptyset$ or there is $q \in G$ s.t. $\forall r \in E \ r \perp q$.

(iv) If $p \in G$ and E is dense below p, then $G \cap E \neq \emptyset$.

Proof of the syntactic forcing theorem

Equality

$p \Vdash^* \tau_0 = \tau_1 \;:\Longleftrightarrow\; \forall (\pi_0, s_0) \in \tau_0$
$\{q \leq p ;\; q \leq s_0 \longrightarrow \exists (\pi_1, s_1) \in \tau_1 \; (q \leq s_1 \wedge q \Vdash^* \pi_0 = \pi_1)\}$
is dense below p and $\forall (\pi_1, s_1) \in \tau_1$
$\{q \leq p ;\; q \leq s_1 \longrightarrow \exists (\pi_0, s_0) \in \tau_0 \; (q \leq s_0 \wedge q \Vdash^* \pi_0 = \pi_1)\}$
is dense below p.

The SynFT for $=$ can be proved by induction on Name_α.

Element

$p \Vdash^* \tau_0 \in \tau_1 \;:\Longleftrightarrow$
$\{q ;\; \exists (\pi, s) \in \tau_1 \; (q \leq s \wedge q \Vdash^* \pi = \tau_0)\}$
is dense below p.

The SynFT for $\in$ can be proved once SynFT for $=$ is established.

$p \Vdash^* \varphi(\vec{\tau}) \wedge \psi(\vec{\tau}) \;:\Longleftrightarrow\; p \Vdash^* \varphi(\vec{\tau})$ and $p \Vdash^* \psi(\vec{\tau})$
$p \Vdash^* \neg\varphi(\vec{\tau}) \;:\Longleftrightarrow\; \forall q \leq p \;\; q \nVdash^* \varphi(\vec{\tau})$
$p \Vdash^* \exists x \varphi(x, \vec{\tau}) \;:\Longleftrightarrow\; \{r ;\; \exists \sigma \;\; r \Vdash^* \varphi(\sigma, \vec{\tau})\}$
is dense below p

Rest is induction by formula complexity.

Let's do $\neg$ and leave rest to ES#3.
Assume SynFT is proved for φ and prove it for $\neg\varphi$.

$\Rightarrow$ Suppose $M[G] \models \neg\varphi$. Consider
$$D := \{p ;\; p \Vdash^* \varphi \text{ or } p \Vdash^* \neg\varphi\}.$$
By def. of $\Vdash^* \neg$, this is dense. Find $p \in G \cap D$.
By assumption $p \nVdash^* \varphi$, so $p \Vdash^* \neg\varphi$.

$\Leftarrow$ Let $p \in G$ s.t. $p \Vdash^* \neg\varphi$. By def. $\overbrace{\forall q \leq p \;\; q \nVdash^* \varphi}^{(*)}$.
If $M[G] \not\models \neg\varphi$, then $M[G] \models \varphi$. By IH, find $r \in G$
$r \Vdash^* \varphi$. So $q \leq r, p$, $q \in G$. By Lemma $q \Vdash^* \varphi$,
contradiction to $(*)$. qed