X Tenth Lecture
FORCING & THE CONTINUUM HYPOTHESIS

Tuesday, 25 February 2025

RECAP

Lecture IX

NAMES

Idea: Think of elements of $\mathbb{P}$ as "truth values" for the von Neumann construction.

$$\mathrm{Name}_0^{\mathbb{P}} := \emptyset$$
$$\mathrm{Name}_{\alpha+1}^{\mathbb{P}} := \{\tau ; \tau \subseteq \mathrm{Name}_\alpha \times \mathbb{P}\}$$
$$\mathrm{Name}_\lambda^{\mathbb{P}} := \bigcup_{\alpha<\lambda} \mathrm{Name}_\alpha^{\mathbb{P}}$$

Then $\mathrm{Name}^{\mathbb{P}} := \bigcup_{\alpha \in \mathrm{Ord}} \mathrm{Name}_\alpha^{\mathbb{P}}$ is the proper class of all names.

INTERPRETATION

$\tau_p := \{(\emptyset, p)\}$

$\tau_{pq} := \{(\tau_p, q)\}$

If $F \subseteq \mathbb{P}$, we interpret a $\mathbb{P}$-name τ as follows:

$$\mathrm{val}(\tau, F) := \{\mathrm{val}(\sigma, F) ; \exists p \in F \ (\sigma, p) \in \tau\}$$

Def. The (generic) extension for any ctm M and any $F \subseteq \mathbb{P}$ whose $\mathbb{P} \in M$ is

$$M[F] := \{\mathrm{val}(\tau, F) ; \tau \in \mathrm{Name}^{\mathbb{P}} \cap M\}$$

Obviously $M[F]$ is a countable set with $\emptyset \in M[F]$ (Example ①).

Also by definition, $M[F]$ is transitive.

Note: $M[F] \models$ Extensionality + Foundation.

Need to show

① $M \subseteq M[F]$

② $F \in M[F]$

③ $M[F] \models \mathrm{ZFC}$

④ $M[F]$ is minimal.

$M[F]$ is transitive

Also:

$\mathrm{Ord} \cap M[F] = \mathrm{Ord} \cap M$.

One more property of the generic extension:

$$\boxed{\mathrm{Ord} \cap M = \mathrm{Ord} \cap M[F]}$$

Lemma If τ is a name, then
$$\mathrm{rank}(\mathrm{val}(\tau, F)) \leq \mathrm{rank}(\tau)$$

Both von Neumann rank.

Proof. Induction.

Corollary $\mathrm{Ord} \cap M = \mathrm{Ord} \cap M[F]$.

Proof. $\subseteq$ is just canonical names.

If $\alpha = \mathrm{val}(\tau, F)$, then
$$\alpha = \mathrm{rank}(\alpha) \leq \mathrm{rank}(\tau) \in \mathrm{Ord} \cap M.$$

q.e.d.

So: $M[F]$ is (potentially) fatter, but not higher.

Further Recap

We proved $M[G] \models$ Extensionality + Foundation + Pairing + Union.

Homework: Think about why power set is not so easy ...

"Union" proof was:
collect all natural candidates of names for elements and assign the natural values.

Problem: If you try to do this for powerset, neither the "natural candidates for names" nor the "natural values" are obvious. It'll turn out that they are obvious in the end, but that requires some assistance.

Note: Separation & Replacement are even worse.

One remaining easy axiom: AC.

By WOT, AC $\iff \forall x \exists \alpha \exists i \quad \underbrace{i : x \longrightarrow \alpha}_{\text{injection}}$

If $x \in M[F]$, then there $\sigma \in$ Name s.t.

$$x = \mathrm{val}(\sigma, F).$$

I write $\mathrm{dom}(\sigma) := \{\tau \,;\, \exists p\ (\tau, p) \in \sigma\}$.

Consider $\mathrm{dom}(\sigma)$. In M, I have $i : \mathrm{dom}(\sigma) \longrightarrow \alpha$ for some ordinal α.

In $M[F]$, define

$$y \longmapsto \min\{ i(\tau) \,;\, \begin{array}{l} \mathrm{val}(\tau, F) = y \\ \tau \in \mathrm{dom}(\sigma) \end{array} \}$$

Call this i^*.

Then $i^* : x \longrightarrow \alpha$ is an injection.

So, AC holds in $M[F]$.

FORCING

Fix M ctm of ZFC, $\mathbb{P} \in M$ forcing poset.

We call the language

$$\mathcal{L}^{M}_{\text{Forcing}} := \mathcal{L}_{\in} \cup \{\tau \,;\, \mathbb{P}\text{-nam}\}$$

the FORCING LANGUAGE (OVER M).

<u>Interpretation</u> If G is a $\mathbb{P}$-generic over M and φ is in the forcing language, we say

$$M[G] \models \varphi$$
$$\iff M[G], v \models \varphi$$

where $v(\tau) := \mathrm{val}(\tau, G)$.

SEMANTIC FORCING PREDICATE:

$$p \Vdash_{M,\mathbb{P}} \varphi \;:\iff\; \forall G \ \mathbb{P}\text{-generic}/M \text{ with } p \in G \quad M[G] \models \varphi.$$

"p forces φ"

[We often omit $M, \mathbb{P}$.]

<u>Two theorems at the heart of forcing</u>

(1) <u>Forcing Theorem</u> If G is $\mathbb{P}$-generic over M, then

$$M[G] \models \varphi \iff \exists p \in G \ \ p \Vdash \varphi$$

(2) <u>Definability Theorem</u> "$p \Vdash_M \varphi$" is absolutely definable for trs models containing M.

We are going to do the following:

(a) Define the syntactic forcing predicate $\Vdash^*$ that is absolutely definable.

(b) Prove the FT for $\Vdash^*$

(c) Derive that $\Vdash \Leftrightarrow \Vdash^*$.

[Lectures $\overline{\text{XI}}$ & $\overline{\text{XII}}$.]

Observation (under the assumption of FT):

① If $q \leq p$ and $p \Vdash \varphi$, then $q \Vdash \varphi$.

② If $p \Vdash \exists x\, \varphi(x)$, then there is a name τ and $q \leq p$ s.t. $q \Vdash \varphi(\tau)$.

[If $p \Vdash \exists x\, \varphi(x)$ and $p \in G$, then by def. $M[G] \models \exists x\, \varphi(x)$, so there is τ name s.t. $M[G] \models \varphi(\tau)$. By FT, find $r \in G$ s.t. $r \Vdash \varphi(\tau)$.
Find $q \leq p, r$, $q \in G$. By ① $q \Vdash \varphi(\tau)$.]

Proof of Separation in $M[G]$:

Let $\varphi(x, x_1, \dots, x_n)$ be an $\mathcal{L}_G$-formula and $x \in M[G]$.

Need name for

$$A := \{ z \in x ;\ M[G] \models \varphi(z, \cancel{\vec{x}}) \}$$

[For readability, drop parameters!]

Fix σ s.t. $\mathrm{val}(\sigma, G) = x$.

Define: $\rho := \{ (\pi, p) ;\ \pi \in \mathrm{dom}(\sigma) \wedge p \Vdash \pi \in \sigma \wedge \varphi(\pi) \}$

By DT, ρ is a name in M.

Claim: $\mathrm{val}(\rho, G) = A$.

"$\subseteq$". If $z \in \mathrm{val}(\rho, G)$, then there is $(\pi, p) \in \rho$ s.t. $z = \mathrm{val}(\pi, G)$, $p \in G$.

$(\pi, p) \in \rho \implies \pi \in \mathrm{dom}(\sigma)$, $p \Vdash \pi \in \sigma \wedge \varphi(\pi)$.

$M[G] \models \pi \in \sigma \wedge \varphi(\pi) \implies M[G] \models z \in x \wedge \varphi(z) \implies z \in A$.

"$\supseteq$" If $z \in A \implies z \in x \implies$ there is $\pi \in \mathrm{dom}(\sigma)$, $z = \mathrm{val}(\pi, G)$

$+$ $M[G] \models \varphi(z)$

$\overset{FT}{\implies} \exists p \in G \ \ p \Vdash \varphi(\pi)$

Also there is $q \in G$ s.t. $q \Vdash \pi \in \sigma$.

Find $r \in G$, $r \leq p, q$ s.t.! $r \Vdash \pi \in \sigma \wedge \varphi(\pi)$

Then $(\pi, r) \in \rho$ by def., so

$z = \mathrm{val}(\pi, G) \in \mathrm{val}(\rho, G)$. q.e.d.