Forcing & the Continuum Hypothesis

Lecture VIII

18 February 2025

Cohen

(*) $\boxed{\forall T \subseteq ZFC \text{ finite } \exists T^* \subseteq ZFC \text{ finite s.t. if } M \text{ ctm of } T^*, \text{ then there is } N \supseteq M \text{ ctm of } T + \neg CH}$

We have seen (ES#1) that (*) implies $Con(ZFC) \implies Con(ZFC + \neg CH)$.

SIMPLIFIED: If M ctm of ZFC, then there is $N \supseteq M$ ctm of $ZFC + \neg CH$

Idea: If M ctm of ZFC
$\alpha := \omega_1^M$; $\beta := \omega_2^M$; $f : \beta \xrightarrow[\text{injection}]{} \mathcal{P}(\omega)$
Force N s.t. $f \in N$ and $M \subseteq N$.

Observe that there is a ctm N s.t. $f \in N$ & $M \subseteq N$:
$M \cup tcl(\{f\})$ is transitive + countable
Thus LSM gives N transitive countable with $M \cup tcl(\{f\}) \subseteq N$.
Know $N \models \exists g : \beta \to \mathcal{P}(\omega)$ injection,
but no clue what $\aleph_1$ & $\aleph_2$ in N are.

How do we control what we add?

$(\mathbb{P}, \leq, \mathbb{1})$ is called a forcing poset or forcing if it is a partial order and $\mathbb{1} \in \mathbb{P}$ and $\mathbb{1}$ is the largest element.

Elements of $\mathbb{P}$ are called conditions.

$p \leq q$ is interpreted as
"p is stronger than q"
"q is weaker than p".

NOTE UNCONVENTIONAL

"JERUSALEM CONVENTION"
Interpret $p \leq q$ as
"q is stronger than p"

p and q are compatible if there is $r \leq p, q$.

p, q incompatible if not compatible
$\iff p \perp q$.

$A \subseteq \mathbb{P}$ is an antichain if any two distinct elts of A are incompatible

$D \subseteq \mathbb{P}$ is dense if
$\forall p \in \mathbb{P} \, \exists q \leq p \, (q \in D)$.

Saharon Shelah
שַׂהַרֹן שֶׁלַח

Shelah in 2005

Born	July 3, 1945 (age 79) Jerusalem, British Mandate for Palestine (now Israel)
Alma mater	Tel Aviv University (B.Sc) Hebrew University (M.Sc., Ph.D.)
Known for	Proper Forcing, PCF theory, Sauer–Shelah lemma, Shelah cardinal

$F \subseteq \mathbb{P}$ is called a filter if

(a) $\forall p, q \in F \ \exists r \in F \ \ r \leq p, q$

(b) $\forall p \in F \ \forall q \in \mathbb{P} \ \ q \geq p \implies q \in F$

Filter cannot contain incompatible elements.

If F only has property (a), we call it a filter base and then

$$\{p \in \mathbb{P};\ \exists q \in F\ (q \leq p)\}$$

is the filter generated from F.

If $\mathcal{D}$ is a family of dense sets, then F is called $\mathcal{D}$-generic if F is a filter and $\forall D \in \mathcal{D}\ \ D \cap F \neq \emptyset$.

<u>Theorem</u> If $\mathcal{D}$ is countable, then there is a $\mathcal{D}$-generic filter.

<u>Proof.</u> Let $\mathcal{D} = \{D_n;\ n \in \mathbb{N}\}$. Pick $p_0 \in D_0$ arbitrarily and define by recursion p_{i+1} by picking some $q \leq p_i$ with $q \in D_{i+1}$. Then $\{p_i;\ i \in \mathbb{N}\}$ is a filter base, so let F be the filter generated by it. Then it's $\mathcal{D}$-generic by construction.

q.e.d.

Main example Fix any sets X & Y.

$$\mathbb{P} := \mathrm{Fn}(X,Y) := \{p \,;\, p \text{ is a finite function with } \mathrm{dom}(p) \subseteq X \text{ and } \mathrm{ran}(p) \subseteq Y\}$$

$$p \leq q :\iff p \supseteq q.$$

$$\mathbb{1} := \emptyset$$

What does $p \perp q$ mean?

$$p \perp q \iff \exists x \in X \;\; x \in \mathrm{dom}(p) \cap \mathrm{dom}(q) \;\&\; p(x) \neq q(x).$$

Lemma 1 If $F \subseteq \mathbb{P}$ is a filter, then $\bigcup F$ is a function.

Lemma 2 $D_x := \{p \in \mathbb{P} \,;\, x \in \mathrm{dom}(p)\}$ is a dense set. If $\mathcal{D} := \{D_x \,;\, x \in X\}$, and F is $\mathcal{D}$-generic, then $\mathrm{dom}(\bigcup F) = X$.

Proof. If $x \in X$, find $p \in F \cap D_x$, then $x \in \mathrm{dom}(p)$ $\implies x \in \mathrm{dom}(\bigcup F)$. q.e.d.

Example 1 — Cohen Forcing

Paul Cohen

American mathematician

Paul Joseph Cohen was an American mathematician. He is best known for his proofs that the continuum hypothesis and the axiom of choice are independent from Zermelo–Fraenkel set theory, for which he was awarded a Fields Medal. Wikipedia

Born: 2 April 1934, Long Branch, New Jersey, United States

Died: 23 March 2007, Stanford, California, United States

Known for: Cohen forcing; Continuum hypothesis

Fields: Mathematics

$\mathbb{C} := \mathrm{Fn}(\omega, 2)$

If F is $\mathcal{D}$-generic, then

$\bigcup F : \omega \longrightarrow 2$ (by previous slide)

Fix $f : \omega \longrightarrow 2$ and define

$N_f := \{ p \in \mathbb{C} ;\ \exists n\ p(n) \neq f(n) \}$

This is a dense set.

__Lemma 3__ If F is $\mathcal{D} \cup \{N_f\}$-generic, then

$\bigcup F \neq f$.

__Corollary__ There is no F that is

$\mathcal{D} \cup \{ N_f ; f : \omega \longrightarrow 2 \}$-generic.

__However__ if M is a ctm and you consider

$\mathcal{N}_M := \{ N_f ; f \in M \}$, then by the theorem, there is a $\mathcal{D} \cup \mathcal{N}_M$-generic.

And for this F, $\bigcup F \notin M$.

"Cohen real"

EXAMPLE 2. $\mathrm{Fn}(X,Y) =: \mathbb{P}$ as before

$$R_y := \{ p \in \mathbb{P} \,;\, y \in \mathrm{ran}(p) \}$$
$$\mathcal{R} := \{ R_y \,;\, y \in Y \}$$

Lemma 4 If F is $\mathcal{D} \cup \mathcal{R}$-generic, then $\bigcup F : X \longrightarrow Y$ is a surjection.

Corollary If $|Y| > |X|$, then there is no $\mathcal{D} \cup \mathcal{R}$-generic.

However if M ctm and, e.g., $X = \omega$, $Y = \omega_1^M =: \alpha$

then α is a ctble ordinal and thus $\mathcal{D} \cup \mathcal{R}$ is ctble and so by Thm, a $\mathcal{D} \cup \mathcal{R}$-generic exists.

"the collapse forcing of ω_1^M".

EXAMPLE 3.

$$\mathrm{Fn}(X \times Y, 2)$$

Assume X is infinite.

Consider $E_{y,y'} := \{ p \in \mathbb{P} ; \exists x \in X$
$p(x,y) \neq p(x,y') \}$

This is dense for $y \neq y'$.

$$\mathcal{E} := \{ E_{y,y'} ; y \neq y' \in Y \}$$

Lemma 5 If F is $\mathcal{D} \cup \mathcal{E}$-generic, then there is an injection from Y into $\wp(X)$.

Proof. Fix y and define

$$A_y := \{ x \in X ; (\bigcup F)(x,y) = 1 \}$$

$E_{y,y'}$ guarantees that $y \longmapsto A_y$ is a injection.

q.e.d.

Corollary If $|Y| > |\wp(X)|$, then there is no $\mathcal{D} \cup \mathcal{E}$-generic.

However if M ctm of ZFC+CH

$\alpha := \omega_1^M \quad \beta := \omega_2^M$

$\mathrm{Fn}(\omega \times \omega_2^M, 2)$. Because $\beta < \omega_1$, $\mathcal{E}$ is countable and so a $\mathcal{D} \cup \mathcal{E}$-generic exists.