

VII

Seventh Lecture of "Forcing & the Continuum Hypothesis" held on Saturday 15 February 2025 in which we shall prove the Continuum Hypothesis (in the Constructible Universe)

GOAL

$L \models GCH$

$$\forall \alpha \quad 2^{\aleph_\alpha} = \aleph_{\alpha+1}$$



John von Neumann

von Neumann in the 1940s

Member of the United States Atomic Energy Commission

In office

March 15, 1955 - February 8, 1957

President Dwight D. Eisenhower

Preceded by Eugene M. Zuckert

Succeeded by John S. Graham

Personal details

Born Neumann János Lajos

December 28, 1903

Budapest, Kingdom of Hungary

Died February 8, 1957

(aged 53)

Washington, D.C., U.S.

von Neumann hierarchy

$$|V_\omega| = \aleph_0$$

$$|V_{\omega+1}| = 2^{\aleph_0} = \aleph_1$$

$$|V_{\omega+2}| = 2^{\aleph_1} = \aleph_2$$

$$|V_{\omega_1}| = \aleph_{\omega_1}$$

constructible hierarchy

$$|L_\omega| = \aleph_0$$

$$|L_{\omega+1}| = \aleph_1$$

$$|V_{\omega+2}| = \aleph_2$$

$$|L_{\omega_1}| = \aleph_1$$

$$|L_\alpha| = |\alpha|$$



Kurt Gödel (1925)

Kurt Gödel

$$P(\omega) \subseteq V_{\omega+1}$$

PROOF OF POWERSSET

$P(\omega) \cap L \subseteq L_\gamma$ for some $\gamma \rightarrow$ no good upper bound for γ

Idea

If we can give an upper bound for γ ,

then

$$\text{in } L: 2^{\aleph_0} \leq |L_\gamma| = |\gamma|.$$

Thus if ω_1 is a bound for γ , CH follows.

LÖWENHEIM-SKOLEM-MOSTOWSKI

Leopold Löwenheim



Born June 26, 1878
Krefeld, Germany
Died May 5, 1957 (aged 78)
Berlin, Germany
Alma mater University of Berlin,
Technische Universität Berlin

Thoralf Skolem



Born 23 May 1887
Sandnessjøen, Norway
Died 23 March 1963 (aged 75)
Oslo, Norway
Nationality Norwegian
Alma mater Oslo University

Andrzej Mostowski



Mostowski in 1973

Born 1 November 1913
Lemberg, Austria-Hungary
Died 22 August 1975 (aged 61)
Vancouver, British
Columbia, Canada
Nationality Polish
Alma mater University of Warsaw
Known for Löwenheim-Mostowski hierarchy
Mostowski collapse lemma
Mostowski model

LÖWENHEIM-SKOLEM (+Tarski-Vaught)

For any M and $X \subseteq M$, $|X| = \kappa$, there is
 $N < M$
 s.t. $|N| = \kappa$ and $X \subseteq N$.

Mostowski

If M is extensional and $X \subseteq M$ transitive,
 then there is N transitive and a unique
 map $\pi: M \rightarrow N$ (the Mostowski collapse)
 that is an isomorphism between (M, ε) and
 (N, ε) . Furthermore, $\pi \upharpoonright X = \text{id}$, so $X \subseteq N$.

LSM: For any $(M, \varepsilon) \models T^{+Ext}$ and any
 $X \subseteq M$ transitive, there is N transitive
 s.t. $|N| = |X|$ and $N \models T$
 and $X \subseteq N$.

infinite →

THE CONDENSATION SENTENCE

Remember $T_L \subseteq ZF$; false s.t. T_L proves existence and absoluteness of the L -hierarchy.

We call the sentence

$$\sigma := \bigwedge T_L \cup \{ \forall \alpha \exists \beta \beta > \alpha \} \cup \{ V = L \}$$

there is no largest ordinal

the CONDENSATION SENTENCE.

Clearly, $L \models \sigma$.

[Remark: By LRT, there is a model M s.t. $M \models \sigma$.]

Theorem (Condensation Characterization).

If M is transitive set and $M \models \sigma$, then $M = L_\alpha$ for some α .

Proof. Since $M \models T_L$, we have that for all $\alpha \in \text{Ord} \cap M$, $L_\alpha \subseteq M$ [even $L_\alpha \in M$]

Therefore $\bigcup_{\alpha \in \text{Ord} \cap M} L_\alpha \subseteq M$.

Since $M \models V = L$, i.e. $\forall x \exists \alpha x \in L_\alpha$, we get $M \subseteq \bigcup_{\alpha \in \text{Ord} \cap M} L_\alpha$.

But by $M \models$ there is no largest ordinal, we get

$\text{Ord} \cap M = \lambda$ limit. So by definition $M = L_\lambda$.
q.e.d.

GÖDEL'S CONDENSATION LEMMA

(1938)

If $\kappa \leq \aleph$ and $x \in \mathbb{L}$, then $x \in \mathbb{L}_{\kappa^+}$.

↗
cardinal
successor



Gödel around 1926

Born Kurt Friedrich Gödel
April 28, 1906
Brünn, Austria-Hungary
Died January 14, 1978 (aged 71)
Princeton, New Jersey, U.S.

Corollary

$$V = \mathbb{L} \Rightarrow GCH$$

Proof

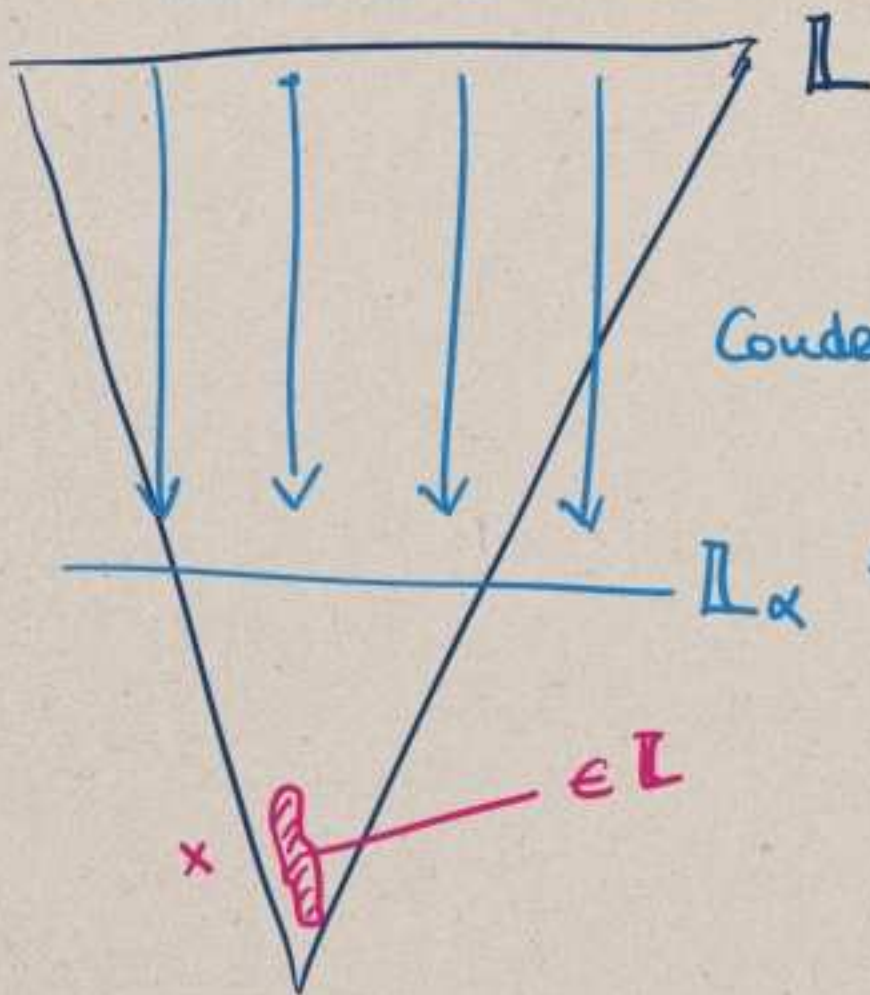
The powerset of κ in $\mathbb{L}$ is

$$\mathcal{P}(\kappa) \cap \mathbb{L} \subseteq \mathbb{L}_{\kappa^+}.$$

$$2^{\aleph_0} = |\mathcal{P}(\aleph_0) \cap \mathbb{L}| \leq |\mathbb{L}_{\aleph_1}| = \aleph_1.$$

↗
in $\mathbb{L}$.

q.e.d.



Condensation.

$\mathbb{L}_\alpha$ s.t. $x \in \mathbb{L}_\alpha$
with
 $|\mathbb{L}_\alpha| = |x|.$

Proof of Gödel's Condensation Lemma

For emotional reasons, let's do $\kappa = \omega$.
Proof is identical for other κ .

Take $x \subseteq \omega$. If x is infinite

$$T_x := \text{tc}(\{x\}) = \{x\} \cup \omega.$$

Clearly, T_x is transitive, countable,
and $x \in T_x$.

Find any γ s.t. $T_x \in \mathbb{L}_\gamma$.

By LRT, find $\mathcal{N} > \gamma$ s.t. σ is absolute
between $\mathbb{L}_\mathcal{N}$ and $\mathbb{L}$.

$$[\text{i.e., } \mathbb{L}_\mathcal{N} \models \sigma.]$$

Know that $x \in T_x \in \mathbb{L}_\mathcal{N}$.

Apply LSM to $\mathbb{L}_\mathcal{N}$ and T_x and find
transitive N s.t. $T_x \subseteq N$, $|N| = |T_x| = \aleph_0$
and $N \models \sigma$.

By Condensation Characterisation, find α s.t.
 $N = \mathbb{L}_\alpha$.

$$T_x \subseteq \mathbb{L}_\alpha, |\mathbb{L}_\alpha| = \aleph_0, \mathbb{L}_\alpha \models \sigma$$

$$\text{Thus } \omega \leq \alpha < \omega_1.$$

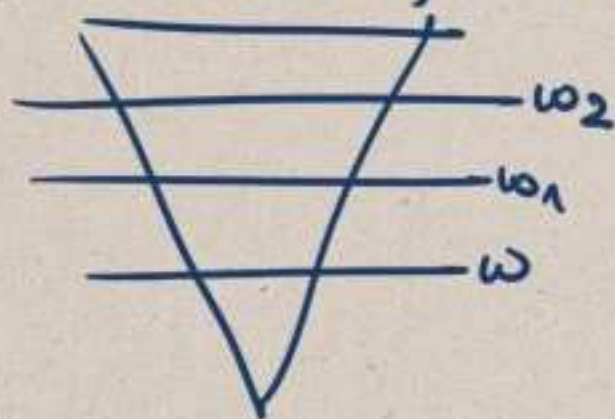
$$x \in T_x \subseteq \mathbb{L}_\alpha \subseteq \mathbb{L}_{\omega_1}. \quad \text{q.e.d.}$$

This means that for each finite $T \subseteq \text{ZFC}$, we find a ctm of $T + \text{GCH}$, viz., L_α for some $\alpha < \omega_1$.

Now: in order to construct a ctm of $\neg \text{CH}$, all we to do is to extend one of these and add witnesses to $\neg \text{CH}$, e.g. $2^{\aleph_0} = \aleph_2$.

How would that work?

M ctm of $T \approx \text{ZFC} + V=L$



In M , we have

$$f: \omega_1^M \rightarrow \kappa(\omega)^M$$

bijection.

$$\omega_1^M = \alpha < \omega_1$$

$$\omega_2^M = \beta < \omega_1$$

$\kappa(\omega)^M$ is countable

Since all of this is ctable, there are many elts of $\kappa(\omega)$ not in M , so take

$$g: \beta \rightarrow \kappa(\omega) \text{ with } g \text{ surjective}$$

and form "the least ZFC-model" $M[g]$.

For the sake of the argument assume that we know what the least ZFC-model means...

Problem:

$$\text{In } M[g] \models \exists g: \beta \rightarrow \mathbb{P}(\omega) \text{ injection} \\ \Rightarrow M[g] \models 2^{\aleph_0} \geq |\beta|.$$

But how do we know that

$$M[g] \models \beta = \aleph_2?$$

If a is a code for α [in the sense of
 $\subseteq \omega$ ES#1 (7)]

and $a \in \text{ran}(g)$, then

$$M[g] \models \alpha \text{ is countable}$$

$\Rightarrow$

$$M[g] \models |\beta| \leq \aleph_1,$$

so we didn't know anything.

Paul Cohen <

American mathematician



Paul Joseph Cohen was an American mathematician. He is best known for his proofs that the continuum hypothesis and the axiom of choice are independent from Zermelo–Fraenkel set theory, for which he was awarded a Fields Medal. [Wikipedia](#)

Born: 2 April 1934, Long Branch, New Jersey, United States

Died: 23 March 2007, Stanford, California, United States

Known for: Cohen forcing; Continuum hypothesis

Fields: Mathematics

Cohen's idea

Not just any g , but a
"generic"

one in such a way that
everything that is true
in $M[g]$ is true for
a reason.

"Forcing Lemma"