VII

Seventh Lecture of "Forcing & the Continuum Hypothesis" held on Saturday 15 February 2025 in which we shall prove the Continuum Hypothesis (in the Constructible Universe)

GOAL $\mathbb{L} \models \mathrm{GCH}$ $\quad \forall\alpha \ 2^{\aleph_\alpha} = \aleph_{\alpha+1}$

von Neumann hierarchy

$|V_\omega| = \aleph_0$

$|V_{\omega+1}| = 2^{\aleph_0} = \beth_1$

$|V_{\omega+2}| = 2^{2^{\aleph_0}} = \beth_2$

$\vdots$

$|V_{\omega_1}| = \beth_{\omega_1}$

$\mathcal{P}(\omega) \subseteq V_{\omega+1}$

constructible hierarchy

$|L_\omega| = \aleph_0$

$|L_{\omega+1}| = \aleph_0$

$|V_{\omega+2}| = \aleph_0$

$|L_{\omega_1}| = \aleph_1$

$\boxed{|L_\alpha| = |\alpha|}$

PROOF OF POWERSET

$\mathcal{P}(\omega) \cap \mathbb{L} \subseteq \mathbb{L}_\gamma$ for some $\gamma \longrightarrow$ no good upper bound for γ.

Idea If we can give an upper bound for γ, then in $\mathbb{L}$: $2^{\aleph_0} \leq |L_\gamma| = |\gamma|$.

Thus if ω_1 is a bound for γ, CH follows.

Löwenheim–Skolem–Mostowski

Löwenheim–Skolem (+ Tarski–Vaught)

For any M and $X \subseteq M$, $|X| = \kappa$, there is

$$N \prec M$$

s.t. $|N| = \kappa$ and $X \subseteq N$.

Mostowski

If M is extensional and $X \subseteq M$ transitive, then there is N transitive and a unique map $\pi : M \longrightarrow N$ (the Mostowski collapse) that is an isomorphism between $(M, \in)$ and $(N, \in)$. Furthermore, $\pi \upharpoonright X = \mathrm{id}$, so $X \subseteq N$.

LSM: For any $(M, \in) \models T^{+\mathrm{Ext}}$ and any $X \subseteq M$ transitive, there is N transitive s.t. $|N| = |X|$ and $N \models T$. and $X \subseteq N$.

infinite

The Condensation Sentence

Remember $T_{\mathbb{L}} \subseteq ZF$; finite s.t. $T_{\mathbb{L}}$ proves existence and absoluteness of the $\mathbb{L}$-hierarchy.

We call the sentence

$$\sigma := \bigwedge T_{\mathbb{L}} \cup \underbrace{\{\forall\alpha\exists\beta\ \beta>\alpha\}}_{\text{there is no largest ordinal}} \cup \{V=\mathbb{L}\}$$

the CONDENSATION SENTENCE.

Clearly, $\mathbb{L} \models \sigma$. [Remark: By LRT, have many β s.t. $\mathbb{L}_\beta \models \sigma$.]

__Theorem__ (Condensation Characterisation).
If M is transitive set and $M \models \sigma$, then $M = L_\alpha$ for some α.

__Proof.__ Since $M \models T_{\mathbb{L}}$, we have that for all $\alpha \in \mathrm{Ord} \cap M$,

$$\mathbb{L}_\alpha \subseteq M \quad [\text{even } \mathbb{L}_\alpha \in M]$$

Therefore $\bigcup_{\alpha \in \mathrm{Ord}\cap M} \mathbb{L}_\alpha \subseteq M$.

Since $M \models V=\mathbb{L}$, i.e. $\forall x \exists\alpha\ x \in \mathbb{L}_\alpha$, we get

$$M \subseteq \bigcup_{\alpha\in\mathrm{Ord}\cap M} \mathbb{L}_\alpha .$$

But by $M \models$ there is no largest ordinal, we get $\mathrm{Ord}\cap M = \lambda$ limit. So by definition $M = \mathbb{L}_\lambda$.

q.e.d.

Gödel's Condensation Lemma

(1938)

If $x \subseteq \kappa$ and $x \in \mathbb{L}$, then $x \in \mathbb{L}_{\kappa^+}$.

cardinal successor

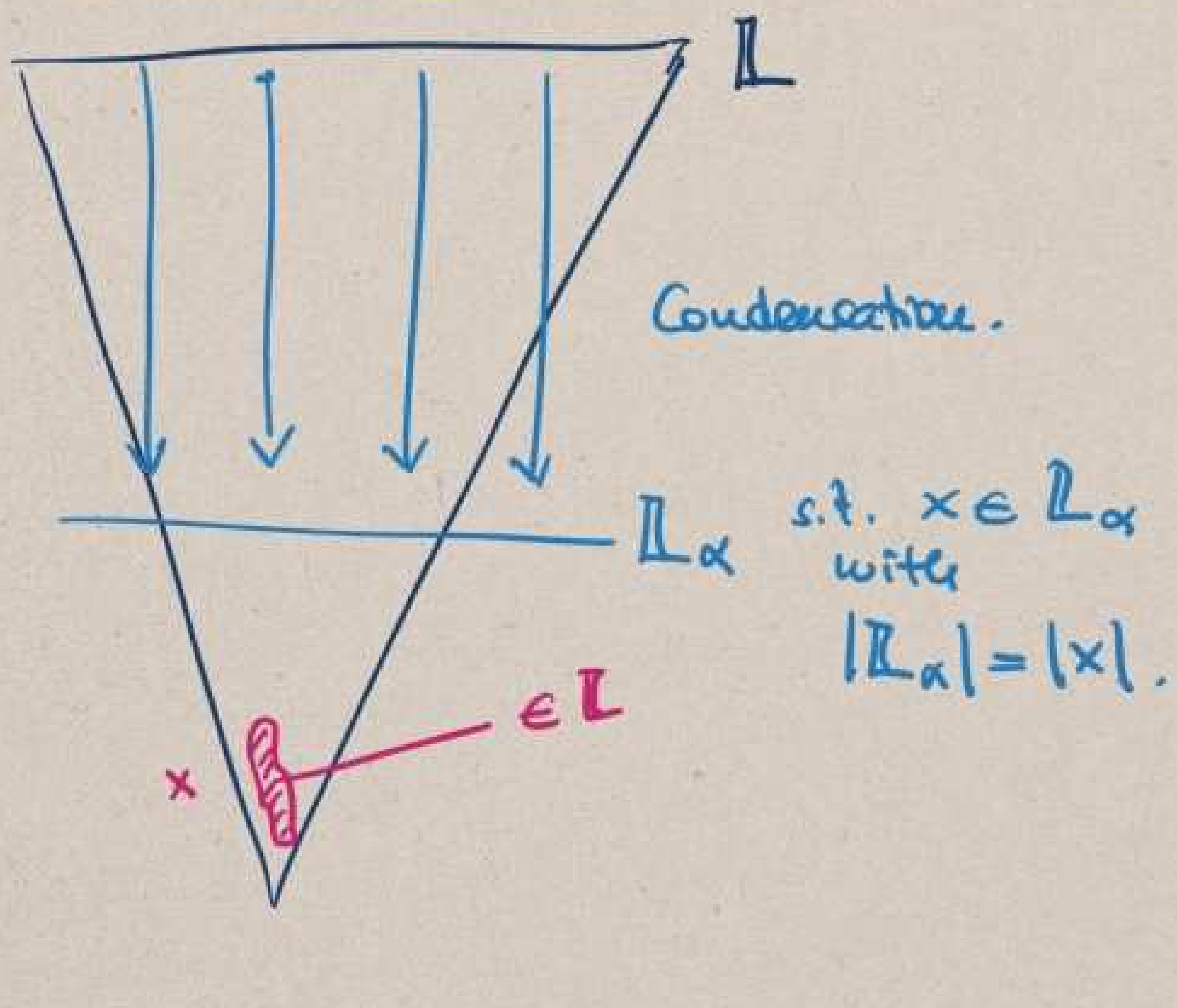
<u>Corollary</u> $\mathbb{V} = \mathbb{L} \Longrightarrow \mathrm{GCH}$

<u>Proof</u> The powerset of κ in $\mathbb{L}$ is $\wp(\kappa) \cap \mathbb{L} \subseteq \mathbb{L}_{\kappa^+}$.

$$2^{\aleph_0} = |\wp(\kappa) \cap \mathbb{L}| \leq |\mathbb{L}_{\kappa^+}| = \kappa^+.$$

in $\mathbb{L}$.

q.e.d.



Proof of Gödel's Condensation Lemma

For emotional reasons, let's do $\kappa = \omega$.
Proof is identical for other κ.

Take $x \subseteq \omega$. If x is infinite
$T_x := \mathrm{tcl}(\{x\}) = \{x\} \cup \omega$.
Clearly, T_x is transitive, countable, and $x \in T_x$.

Find any γ s.t. $T_x \in \mathbb{L}_\gamma$.
By LRT, find $\vartheta > \gamma$ s.t. σ is absolute between $\mathbb{L}_\vartheta$ and $\mathbb{L}$.
[i.e., $\mathbb{L}_\vartheta \models \sigma$.]
Know that $x \in T_x \in \mathbb{L}_\vartheta$.

Apply LSM to $\mathbb{L}_\vartheta$ and T_x and find transitive N s.t. $\boxed{T_x \subseteq N,\ |N| = |T_x| = \aleph_0 \text{ and } N \models \sigma.}$

By Condensation Characterisation, find α s.t.
$N = \mathbb{L}_\alpha$.

$T_x \subseteq L_\alpha$, $|L_\alpha| = \aleph_0$, $L_\alpha \models \sigma$

Thus $\omega \leq \alpha < \omega_1$.

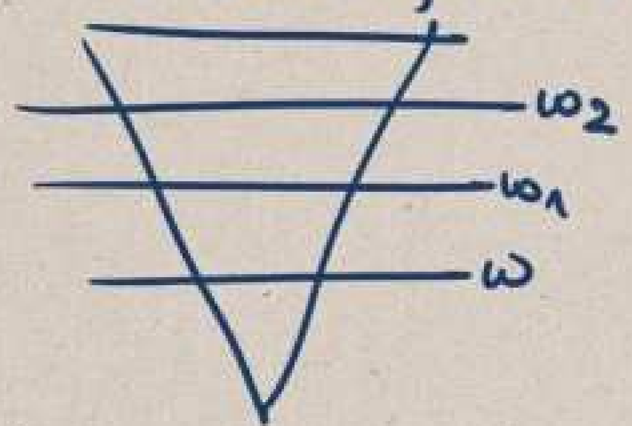
$x \in T_x \subseteq L_\alpha \subseteq L_{\omega_1}$. q.e.d.

This means that for each finite $T \subseteq ZFC$, we find ctm of $T + GCH$, viz., L_α for some $\alpha < \omega_1$.

Now: in order to construct a ctm of $\neg CH$, all we to do is to extend one of these and add witnesses to $\neg CH$, e.g. $2^{\aleph_0} = \aleph_2$.

How would that work?

M ctm of $T \approx ZFC + V{=}L$



In M, we have

$$f : \omega_1^M \longrightarrow \mathcal{P}(\omega)^M$$

bijection.

$\omega_1^M = \alpha < \omega_1$ $\quad \mathcal{P}(\omega)^M$ is countable

$\omega_2^M = \beta < \omega_1$

Since all of this is ctble, there are many elts of $\mathcal{P}(\omega)$ not in M, so take

$$g : \beta \longrightarrow \mathcal{P}(\omega) \text{ with } g \text{ injective}$$

and form "the least ZFC-model" $M[g]$.
??

For the sake of the argument assume that we know what the least ZFC-model means...

Problem:

In $M[G] \models \exists g : \beta \to P(\omega)$ injection

$\Longrightarrow M[G] \models 2^{\aleph_0} \geq |\beta|$.

But how do we know that

$M[G] \models$ "$\beta = \aleph_2$" ?

If $\underset{\subseteq \omega}{a}$ is a code for α [in the sense of ES#1 (7)]

and $a \in \mathrm{ran}(g)$, then

$M[G] \models \alpha$ is countable

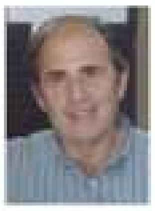
$\Longrightarrow$

$M[G] \models |\beta| \leq \aleph_1$,

so we didn't learn anything.

Paul Cohen

American mathematician



Paul Joseph Cohen was an American mathematician. He is best known for his proofs that the continuum hypothesis and the axiom of choice are independent from Zermelo–Fraenkel set theory, for which he was awarded a Fields Medal. Wikipedia

Born: 2 April 1934, Long Branch, New Jersey, United States

Died: 23 March 2007, Stanford, California, United States

Known for: Cohen forcing; Continuum hypothesis

Fields: Mathematics

Cohen's idea

Not just any g, but a "generic" one in such a way that everything that is true in $M[g]$ is true for a reason.

"Forcing Lemma"