VI

FORCING & THE CONTINUUM HYPOTHESIS

Sixth Lecture — Tuesday, 11 February 2025

Kurt Gödel

Gödel c. 1926

Born	Kurt Friedrich Gödel April 28, 1906 Brünn, Austria-Hungary (now Brno, Czech Republic)
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Gödel 1938:
$\mathrm{Con(ZFC)} \longrightarrow \mathrm{Con(ZFC + GCH)}$

The constructible hierarchy

Fix set X, define for each $\varphi \in \mathrm{Fml}$ and each $p \in X^{<\omega}$ (parameter)

$$D(\varphi, p, X) := \{ w \in X ; \; X \models \varphi(p, w) \}$$

the subset of X defined by φ w/ parameter p

For a sufficiently strong $T \subseteq \mathrm{ZFC}$ finite, we have that T proves that D is an absolute operation.

ES#1

"definable powerset of X"

$$\mathcal{D}(X) := \{ D(\varphi, p, X) ; \; \varphi \in \mathrm{Fml}, p \in X^{<\omega} \}$$

This is absolute for a sufficiently strong theory (use Replacement to get $\mathcal{D}(X)$).

Obvious: $\mathcal{D}(X) \subseteq \wp(X)$.
Also: if X is transitive, then so is $\mathcal{D}(X)$.

$a \in \mathcal{D}(X) \iff \exists \varphi \in \mathrm{Fml} \; \exists p \in X^{<\omega} \; a = D(\varphi, p, X)$

$$L_0 := \emptyset$$
$$L_{\alpha+1} := \mathcal{D}(L_\alpha)$$
$$L_\lambda := \bigcup_{\alpha < \lambda} L_\alpha$$

The constructible hierarchy

There is a finite fragment $T_{\mathbb{L}}$ of ZF [!] that proves that all of the operations occurring in the def. of $\mathbb{L}$, i.e., Fml, $X^{<\omega}$, $\models$, D, $\mathcal{D}$ are well-defined and absolute.

Thus, if M is a transitive model of $T_{\mathbb{L}}$, then

$$\forall \alpha \in \mathrm{Ord} \cap M \;\; \mathbb{L}_\alpha \in M$$
and thus $\mathbb{L}_\alpha \subseteq M$.

So $\displaystyle\bigcup_{\alpha \in \mathrm{Ord} \cap M} \mathbb{L}_\alpha \subseteq M$.

AXIOMS OF ZF

STRUCTURAL AXIOMS

EXTENSIONALITY

$$\forall x \forall y \left(\forall w (w \in x \leftrightarrow w \in y) \longrightarrow x = y \right)$$

FOUNDATION

$$\forall x \left(\exists v (v \in x) \longrightarrow \exists m \left(m \in x \wedge \forall w \neg (w \in m \wedge w \in x) \right) \right)$$

INFINITY

$$\exists i \, \exists e (\forall v \neg v \in e \wedge e \in i) \wedge \forall x \left(x \in i \longrightarrow \exists s \left(s \in i \wedge \forall w (w \in s \leftrightarrow w \in x \vee w = x) \right) \right)$$

FUNCTIONAL AXIOMS

PAIRING

$$\forall x \forall y \exists p \quad \forall w \left(w \in p \leftrightarrow w = x \vee w = y \right)$$

UNION

$$\forall x \exists u \quad \forall w \left(w \in u \leftrightarrow \exists z (z \in x \wedge w \in z) \right)$$

POWERSET

$$\forall x \exists p \quad \forall w \left(w \in p \leftrightarrow \forall v (v \in w \rightarrow v \in x) \right)$$

SEPARATION φ

$$\forall \vec{p} \, \forall x \exists s \quad \forall w \left(w \in s \leftrightarrow w \in x \wedge \varphi(w, \vec{p}) \right)$$

REPLACEMENT φ

$$\forall \vec{p} \quad \forall x \forall y \forall z \left(\varphi(x, y, \vec{p}) \wedge \varphi(x, z, \vec{p}) \rightarrow y = z \right) \longrightarrow \forall x \exists r \; \forall w \left(w \in r \leftrightarrow \exists y \left(y \in x \wedge \varphi(y, w, \vec{p}) \right) \right)$$

THE STRUCTURAL AXIOMS

Extensionality
Foundation
Infinity

In lecture II, we proved Extensionality & Foundation in all transitive structures, so also in $\mathbb{L}$.

Note that ω satisfies the condition of the axiom of infinity, so any M transitive with $\omega \in M$ will satisfy the axiom of infinity.

PAIRING & UNION

$$\forall x \forall y \exists p \quad \forall w \ (w \in p \leftrightarrow w = x \lor w = y)$$
$$\forall x \ \exists u \quad \forall w \ (w \in u \leftrightarrow \exists y \in x \, (w \in y))$$

Since the definitions of pairs and unions

$$z = \{x, y\}$$
$$z = \bigcup x$$

are absolute for transitive models, it's enough to show that $\forall x, y \in \mathbb{L} \ \{x, y\} \in \mathbb{L}$

$$\forall x \in \mathbb{L} \ \bigcup x \in \mathbb{L}.$$

If $x, y \in \mathbb{L}_\alpha$ $\quad \varphi(w, x, y) := w = x \lor w = y.$

$$D(\varphi, (x, y), L_\alpha) = \{w \in L_\alpha \,;\, L_\alpha \models \varphi(w, x, y)\}$$
$$= \{w \in L_\alpha \,;\, L_\alpha \models w = x \lor w = y\}$$
$$= \{x, y\}.$$

Similarly for the union with $\varphi(w, x) := \exists y \, (y \in x \land w \in y)$.

POWERSET

$$\forall x \exists p \ \underbrace{\forall w (w \in p \leftrightarrow w \subseteq x)}_{*}$$

The problem here is that
$*$ is not obviously absolute.
In particular $z = \wp(x)$ is not absolute.

Consider $\mathbb{L}_{\omega+1}$: we have $\omega \in \mathbb{L}_{\omega+1}$ and
$\wp(\omega) \cap \mathbb{L}_{\omega+1}$ is countable.

In $\mathbb{L}_{\omega+2}$, we find

$$\{ a \in \mathbb{L}_{\omega+1} ; a \subseteq \omega \}$$

which is the best possible answer to the Q
What is the power set of ω? that $\mathbb{L}_{\omega+1}$ can give, but unlikely to be the correct answer.

Consider instead $\wp(\omega) \cap \mathbb{L} =: P$ and
define $\Omega := \{ \rho_{\mathbb{L}}(a) ; a \in P \}$.

By Replacement, Ω is a set of ordinals, so find $\alpha > \Omega$. Then

$\rho_{\mathbb{L}}(a) :=$ least α s.t. $a \in \mathbb{L}_{\alpha+1}$.

$$P \subseteq \mathbb{L}_\alpha .$$

Therefore $\underbrace{\{ a \in \mathbb{L}_\alpha ; a \subseteq \omega \}}_{= P} \in \mathbb{L}_{\alpha+1}$, so $P \in \mathbb{L}$.

SEPARATION

$$\forall \vec{p}\, \forall x\, \exists s\, \forall w \left(w \in s \longleftrightarrow \underbrace{w \in x \wedge \varphi(w, \vec{p})}_{\varphi'(w, x, \vec{p})} \right)$$

If $x \in L_\alpha$, then

$$D(\varphi', x, L_\alpha) := \{ w \in L_\alpha ;\ L_\alpha \models \varphi'(w, x, \vec{p}) \}$$

$$= \{ w \in L_\alpha ;\ L_\alpha \models w \in x \wedge \varphi(w, \vec{p}) \}$$

$$\overset{?}{=} \{ w \in \mathbb{L} ;\ \mathbb{L} \models w \in x \wedge \varphi(w, \vec{p}) \}$$

not a problem

this is a problem

If φ is not absolute between L_α and L, this won't work.

LRT to the rescue:

$\forall \varphi\, \forall \alpha\, \exists \vartheta > \alpha$ φ is abs. between L_ϑ and L.

Thus: form $D(\varphi', x, L_\vartheta) =$

$$\{ w \in L_\vartheta ;\ L_\vartheta \models w \in x \wedge \varphi(w, \vec{p}) \}$$

$$\overset{\text{Abs.}}{=} \{ w \in L_\vartheta ;\ L \models w \in x \wedge \varphi(w, \vec{p}) \}.$$

REPLACEMENT

is going to be on ES#2.
The proof is a combination of the ideas from powerset & separation.

Corollary (Minimality).
If T is a transitive model of ZF, then for all $\alpha \in T \cap \mathrm{Ord}$, $\mathbb{L}_\alpha \subseteq T$.

Axiom of Constructibility: $V = L$
$$\forall x \exists \alpha \; x \in L_\alpha .$$

Therefore $\mathbb{L} \models V = L$.

Remark on the Axiom of Choice

Kurt Gödel (1925)

Kurt Gödel

Gödel (1938)

$$\mathrm{Con}(ZF) \longrightarrow \mathrm{Con}(ZFC)$$

Note first that everything we did so far only needed ZF in the universe.

We shall sketch that $\mathbb{L} \models AC$.

In fact, a strong version of AC

GLOBAL CHOICE:

there is an absolutely definable bijective operation between $\mathbb{L}$ and Ord

Sketch: Recursive construction of bijections

$$\pi_\alpha : \mathbb{L}_\alpha \longrightarrow \eta_\alpha \quad \text{some ordinal}$$

s.t. $\beta < \alpha$

$$\pi_\alpha \restriction \mathbb{L}_\beta = \pi_\beta .$$

If λ is limit and π_α is defined for $\alpha < \lambda$, then let $\pi_\lambda(x) := \pi_\alpha(x)$ if $x \in \mathbb{L}_\alpha$.

Suppose $\alpha = \beta + 1$ and π_β is given $\pi_\beta : L_\beta \to \eta_\beta$.

Consider $\mathrm{Fml} \times L_\beta^{<\omega}$. Wellorder it in order type η'_β via the induced π_β-wellorder

Then if $x \in L_\alpha$

$$\pi_\alpha(x) := \begin{cases} \pi_\beta(x) & \text{if } x \in L_\beta \\ \eta_\beta + \xi & \text{if } \xi \text{ is the ordinal corresp. to the least } (\varphi, \vec{p}) \text{ s.t. } x = D(\varphi, \vec{p}, L_\beta) \end{cases}$$