V

FIFTH LECTURE

Forcing & the Continuum Hypothesis

8 February 2025

TODAY: Introduction of the constructible hierarchy

RECAP

Absoluteness is preserved under transfinite recursion

Let F, G, H be three operations.

RECURSION EQUATION $(*)$

$$\begin{cases} R(0,\vec{x}) := F(\vec{x}) \\ R(\alpha+1,\vec{x}) := G(\alpha, R(\alpha,\vec{x}), \vec{x}) \\ R(\lambda,\vec{x}) := H(\lambda, \{R(\alpha,\vec{x}) ; \alpha<\lambda\}, \vec{x}) \end{cases}$$

Note that for F, G, H there is a finite fragment $T_{F,G,H} \subseteq ZFC$ that proves the recursion theorem instance for F, G, H.

$T_{F,G,H} \vdash L1(F,G,H)$

$T_{F,G,H} \vdash L2(F,G,H)$

$T_{F,G,H}$ proves existence of R

Proof of recursion theorem:

Attempts: set functions satisfying $(*)$

L1: All attempts agree on their common domain.

L2: $\forall \alpha \exists r$ attempt $(\alpha,\vec{x}) \in \mathrm{dom}(r)$

$R(\alpha,\vec{x}) := y \quad :\Longleftrightarrow$

$\exists r$ attempt $(\alpha,\vec{x}) \in \mathrm{dom}(r)$ and $r(\alpha,\vec{x}) = y$

CONVENTION We say "T is sufficiently strong" if $T \subseteq ZFC$ is finite and T proves the existence of all relevant operations such that they are absolute for transitive models of T.

Let F, G, H be three operations.

RECURSION EQUATION (*)

$$\begin{cases} R(0,\vec{x}) := F(\vec{x}) \\ R(\alpha+1,\vec{x}) := G(\alpha, R(\alpha,\vec{x}), \vec{x}) \\ R(\lambda,\vec{x}) := H(\lambda, \{R(\alpha,\vec{x}) ; \alpha<\lambda\}, \vec{x}) \end{cases}$$

Theorem If $T \supseteq T_{F,G,H}$ and F, G, H are absolute for transitive models of T, then so is R defined by (*).

PROOF Observe that by assumption, being an "attempt" is absolute for transitive models of T.

Let $M \models T$ be transitive.

(1) To show: If $M \models R(\alpha,\vec{x}) = z$, then $R(\alpha,\vec{x}) = z$.

$\Downarrow$

$$M \models \exists r \underbrace{\boxed{\begin{array}{l} r \text{ is an attempt and} \\ r(\alpha,\vec{x}) = z. \end{array}}}_{\exists r \text{ (absolute)}} \quad \text{absolute}$$

$\Rightarrow$ upwards absolute

Thus: there is r s.t. r is attempt and $r(\alpha,\vec{x}) = z$.

So $R(\alpha,\vec{x}) = z$.

(2) Other direction. Assume r is attempt with $r(\alpha,\vec{x}) = z$.

Since $T_{F,G,H} \vdash L2(F,G,H)$, we have

$M \models \exists r'$ $\underline{r' \text{ is attempt } \& \ (\alpha,\vec{x}) \in \mathrm{dom}(r')}$

absolute $\Rightarrow$ r' is a real attempt

By L1, $r'(\alpha,\vec{x}) = r(\alpha,\vec{x})$. $\Rightarrow M \models R(\alpha,\vec{x}) = z$. qed

Note This uses the fact that Δ_1 concept are absolute.

Def. A property is called Δ_1^T if it's both Σ_1^T and Π_1^T.

Observe Δ_1^T concepts are absolute
[upwards from Σ_1 & downwards from Π_1]

Typical applications

BOUNDING A QUANTIFIER BY OPERATION

Let F be an operation and T strong enough to prove F is operation & absolute

$$T \vdash \forall x \exists z \; F(x) = z$$
$$T \vdash \forall x \forall z \forall z' \; F(x) = z \land F(x) = z' \rightarrow z = z'$$

Then the quantifiers

$$\exists y \in F(x) \text{ and } \forall y \in F(x)$$

preserve absoluteness.

$$\exists y \in F(x)\, \psi \iff \exists z\, (\underbrace{\underbrace{z = F(x) \land \exists y \in z\, \psi}_{\text{absolute}})}_{\text{upwards absolute}}$$

$$\iff \forall z\, (\underbrace{\underbrace{z = F(x) \rightarrow \exists y \in z\, \psi}_{\text{absolute}})}_{\text{downwards absolute}}$$

APPLICATIONS

(1) Encode formulas as elements of $\omega^{<\omega}$

$\in$	$=$	$($	$)$	$\wedge$	$\vee$	$\neg$	$\exists$	$\forall$	v_0	v_1	v_2	...
0	1	2	3	4	5	6	7	8	9	10	11	...

$$\forall v_0 \exists v_1 \neg v_0 \in v_1$$

$$(8, 9, 7, 10, 6, 9, 0, 10)$$

$\mathrm{Fml} \subseteq \omega^{<\omega}$

So, Fml is absolute for some (sufficiently strong) finite fragment of ZFC. ES#1

(2) If X is any set then

$$\text{“}X \models \varphi\text{”} \quad [\text{“}(X, \in) \models \varphi\text{”}]$$

is defined by the usual (Tarski) recursion and thus also absolute.

ES#1

The constructible hierarchy

Fix set X, define for each $\varphi \in \mathrm{Fml}$ and each $p \in X^{<\omega}$
parameter

$$D(\varphi, p, X) := \{ w \in X ; X \models \varphi(w, p) \}$$

the subset of X defined by φ w/ parameters p

For a sufficiently strong $T \subseteq$ ZFC finite, we have that T proves that D is an absolute operation.

ES#1

"definable powerset of X"

$$\mathcal{D}(X) := \{ D(\varphi, p, X) ; \varphi \in \mathrm{Fml}, p \in X^{<\omega} \}$$

This is absolute for a sufficiently strong theory (use Replacement to get $\mathcal{D}(X)$).

Obvious: $\mathcal{D}(X) \subseteq \wp(X)$.
Also: If X is transitive, then so is $\mathcal{D}(X)$.

$a \in \mathcal{D}(X) \iff \exists \varphi \in \mathrm{Fml}\ \exists p \in X^{<\omega}\ a = D(\varphi, p, X)$

$$L_0 := \emptyset$$
$$L_{\alpha+1} := \mathcal{D}(L_\alpha)$$
$$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$$

The constructible hierarchy

$$L_0 := \emptyset$$
$$L_{\alpha+1} := \mathcal{D}(L_\alpha)$$
$$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$$

We usually write
$$L := \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha$$

<u>Claim</u> L is a hierarchy in the sense of L III.

ES#1

By closure of absoluteness under transfinite recursion, the L-hierarchy is absolute for trs models of $T \subseteq ZFC$ where T is strong enough to prove that it exists

i.e., if $M \models T$ transitive and $\alpha \in \mathrm{Ord}\cap M$ and $M \models X = L_\alpha$, then $X = L_\alpha$.

So $\bigcup_{\alpha\in\mathrm{Ord}\cap M} L_\alpha \subseteq M$.

From Lecture III:

<u>Def.</u> We call an assignment $\alpha \longmapsto Z_\alpha$ a HIERARCHY, if

(i) Z_α is a trs set,
(ii) $\mathrm{Ord}\cap Z_\alpha = \alpha$,
(iii) $\alpha<\beta \Rightarrow Z_\alpha \subseteq Z_\beta$, and
(iv) λ limit $\rightarrow Z_\lambda = \bigcup_{\alpha<\lambda} Z_\alpha$.

If $\{Z_\alpha ; \alpha\in\mathrm{Ord}\}$ is a hierarchy, we can define $Z := \bigcup_{\alpha\in\mathrm{Ord}} Z_\alpha$. This is a proper class with $\mathrm{Ord}\subseteq Z$.

and $\rho_Z(x) := \min\{\alpha ; x\in Z_\alpha\}$ a notion of Z-rank.

Main Thm of L VI

$L \models ZF$ & if $M \models ZF$ trs, then
$$\bigcup_{\alpha \in \mathrm{Ord} \cap M} L_\alpha \models ZF.$$

[Minimal ZF-model]

Some first idea of what the L-hierarchy is like

Clearly, by ind. $L_\alpha \subseteq V_\alpha$.

$L_0 := \emptyset$

$L_{\alpha+1} := \mathcal{D}(L_\alpha)$

$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$

Clearly, for $n \in \omega$ $L_n = V_n$

$\Longrightarrow L_\omega = V_\omega$

$$L_{\alpha+1} := \bigcup_{\varphi \in \mathrm{Fml}} \bigcup_{p \in L_\alpha^{<\omega}} \{D(\varphi, p, L_\alpha)\}$$

If $\alpha \geq \omega$, then $|L_{\alpha+1}| \leq \aleph_0 \cdot |L_\alpha^{<\omega}|$
$= \aleph_0 \cdot |L_\alpha|$

Thus $|L_\alpha| = |L_{\alpha+1}|$.

Therefore $\alpha < \omega_1$ $|L_\alpha| = \aleph_0$ & $|L_{\omega_1}| = \aleph_1$.

This means: $V_{\omega+1} \neq L_{\omega+1}$

size $2^{\aleph_0}$ size $\aleph_0$

AXIOM OF CONSTRUCTI-BILITY

Note: This does not mean $V \neq L$.

$V = L :=$ $\forall x \exists \alpha \; x \in L_\alpha$