IX

Ninth Lecture of

FORCING & THE CONTINUUM HYPOTHESIS

Saturday, 22 February 2025

RECAP M ctm of ZFC [or a sufficiently large finite fragment]

$\mathbb{P} \in M$: dense / filter / $\mathcal{D}$-generic

Thm If $\mathcal{D}$ is countable, then there is a $\mathcal{D}$-generic.

Example 1 $\mathrm{Fn}(\omega, 2)$ produces a new function $f : \omega \longrightarrow 2$

Cohen forcing

Example 2 $\mathrm{Fn}(X, Y)$ produces a surjection $f : X \longrightarrow Y$

$\mathrm{Fn}(\omega, Y)$ COLLAPSE of Y

Example 3 $\mathrm{Fn}(X \times Y, 2)$ produces an injection $f : Y \longrightarrow \wp(X)$

If M is ctm of ZFC, then $\mathcal{D}_M := \{D \subseteq \mathbb{P} \text{ dense}; D \in M\}$ is countable, so by Thm, we have a $\mathcal{D}_M$-generic.

Def. We say F is $\mathbb{P}$-generic over M if it is $\mathcal{D}_M$-generic.

These always exist if M is ctm.

GOAL Build an extension $M[G]$ s.t. $M \subseteq M[G]$, $M[G]$ ctm of ZFC, $G \in M[G]$ & $M[G]$ is minimal.

NAMES

Idea: Think of elements of $\mathbb{P}$ as "truth values" for the von Neumann construction.

$$\mathrm{Name}_0^{\mathbb{P}} := \emptyset$$

$$\mathrm{Name}_{\alpha+1}^{\mathbb{P}} := \{\tau \,;\, \tau \subseteq \mathrm{Name}_\alpha \times \mathbb{P}\}$$

$$\mathrm{Name}_\lambda^{\mathbb{P}} := \bigcup_{\alpha<\lambda} \mathrm{Name}_\alpha^{\mathbb{P}}$$

Then $\mathrm{Name}^{\mathbb{P}} := \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{Name}_\alpha^{\mathbb{P}}$ is the proper class of all names.

[Consider: $\mathbb{P} = \{\mathbb{0},\mathbb{1}\}$
Then $\mathrm{Name}^{\mathbb{P}} \cong V$.]

Since this is a recursive definition using absolute concepts, being a name is absolute for transitive models:

$$\{\tau \,;\, M \models \tau \text{ is a } \mathbb{P}\text{-name}\} = \mathrm{Name}^{\mathbb{P}} \cap M.$$

EXAMPLES

$$\text{Name}^{\mathbb{P}}_0 := \emptyset$$
$$\text{Name}^{\mathbb{P}}_{\alpha+1} := \{\tau ; \tau \subseteq \text{Name}_\alpha \times \mathbb{P}\}$$
$$\text{Name}^{\mathbb{P}}_\lambda := \bigcup_{\alpha<\lambda} \text{Name}^{\mathbb{P}}_\alpha$$

$$\emptyset \in \text{Name}^{\mathbb{P}}_1$$

$$\tau_p := \{(\emptyset, p)\} \in \text{Name}^{\mathbb{P}}_2$$

"The name for a set that contains $\emptyset$ with value p".

$$\tau_{pq} := \{(\tau_p, q)\}$$

"The name for a set that contains whatever τ_p describes with value q".

INTERPRETATION

$\tau_p := \{(\emptyset, p)\}$

$\tau_{pq} := \{(\tau_p, q)\}$

If $F \subseteq \mathbb{P}$, we interpret a $\mathbb{P}$-name τ as follows:

$$\mathrm{val}(\tau, F) := \{\mathrm{val}(\sigma, F) ; \exists p \in F \ (\sigma, p) \in \tau\}$$

Important. This is a recursive definition.

Thus: the valuation is absolute for the models containing τ and F.

Examples ① $\emptyset$; clearly $\mathrm{val}(\emptyset, F) = \emptyset$.

② τ_p; $\mathrm{val}(\tau_p, F) := \begin{cases} \{\emptyset\} & \text{if } p \in F \\ \emptyset & \text{if } p \notin F \end{cases}$

③ τ_{pq}; $\mathrm{val}(\tau_{pq}, F) := \begin{cases} \emptyset & \text{if } q \notin F \\ \{\{\emptyset\}\} & \text{if } q \in F \ \& \ p \in F \\ \{\emptyset\} & \text{if } q \in F \ \& \ p \notin F \end{cases}$

The relationship between p & q affects these possibilities.

E.g., if $q \leq p$ & F is filter, then $\{\emptyset\}$ is impossible.

E.g., if $q = \mathbb{1}$ & F is a nonempty filter, then $\emptyset$ is impossible.

E.g., if $p \perp q$ and F is filter, then $\{\{\emptyset\}\}$ is impossible.

Def. The (generic) extension for any ctm M and any $F \subseteq \mathbb{P}$ whose $\mathbb{P} \in M$ is

$$M[F] := \{ val(\tau, F) \, ; \, \tau \in Name^{\mathbb{P}} \cap M \}$$

Obviously $M[F]$ is a countable set with $\emptyset \in M[F]$ (Example ①).
Also by definition, $M[F]$ is transitive.

Note : $M[F] \models$ Extensionality + Foundation.

Need to show

① $M \subseteq M[F]$
② $F \in M[F]$
③ $M[F] \models ZFC$
④ $M[F]$ is minimal.

CANONICAL NAMES

Let $x \in M$. Define by recursion the canonical name for x by

$$\check{x} := \{(\check{y}, \mathbb{1}) ; y \in x\}$$

Pronounced: "x check"

$\widecheck{p(x \cup y \setminus z)}$

[Notation where we form a canonical name of a more complex expression.]

Lemma 1 $\mathrm{val}(\check{x}, F) = x$ if $\mathbb{1} \in F$.

Proof Induction.

Corollary $M \subseteq M[F]$ if $\mathbb{1} \in F$.

Alternative construction of canonical names w/o $\mathbb{1}$ on ES#2.

$$\Gamma := \{(\check{p}, p) ; p \in \mathbb{P}\}$$

Lemma 2 $\mathrm{val}(\Gamma, F) = F$

Proof
$$\begin{aligned} \mathrm{val}(\Gamma, F) &= \{\mathrm{val}(\check{p}, F) ; p \in F\} \\ &\overset{L1}{=} \{p ; p \in F\} = F. \end{aligned}$$

Corollary $F \in M[F]$.

Remark If N ctm with $M \subseteq N$ and $F \in N$, then $M[F] \subseteq N$.
[By absoluteness of $\mathrm{val}(\tau, F)$.]

Warm-up Suppose $\sigma, \tau \in \mathrm{Name}^{\mathbb{P}}$.

Define $\mathrm{up}(\sigma,\tau) := \{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\}$.

PAIRING

$$\mathrm{val}(\mathrm{up}(\sigma,\tau), F) = \{\mathrm{val}(\sigma, F), \mathrm{val}(\tau, F)\}$$

unordered pair

Corollary $M[F] \models$ Pairing. [if $\mathbb{1} \in F$.]

UNION If τ is a name, define

$$u_\tau := \{(\sigma', r) ; \exists \sigma, p, q \text{ s.t. } (\sigma, p) \in \tau, (\sigma', q) \in \sigma, r \leq p, q\}$$

Corollary $M[F] \models$ Union. [if F is a filter.]

Claim $\mathrm{val}(u_\tau, F) = \bigcup \mathrm{val}(\tau, F)$ if F filter

Proof. $\Longrightarrow$ $z \in \mathrm{val}(u_\tau, F) \Rightarrow z = \mathrm{val}(\sigma', F)$ for some $(\sigma', r) \in u_\tau$ with $r \in F$

$\Rightarrow \exists \sigma, p, q$ $(\sigma, p) \in \tau$, $(\sigma', q) \in \sigma$, $r \leq p, q$

$\Rightarrow p, q \in F$

$\Rightarrow \mathrm{val}(\sigma, F) \in \mathrm{val}(\tau, F)$

$z = \mathrm{val}(\sigma', F) \in \mathrm{val}(\sigma, F)$

$\Rightarrow z \in \bigcup \mathrm{val}(\tau, F)$.

$\Longleftarrow$ $z \in \bigcup \mathrm{val}(\tau, F)$

$\Rightarrow \exists y$ $z \in y \in \mathrm{val}(\tau, F)$

$(\sigma, p) \in \tau$ with $p \in F$

$(\sigma', q) \in \sigma$ with $q \in F$

$\Rightarrow$ since F filter, find $r \leq p, q$, $r \in F$

Then $(\sigma', r) \in u_\tau \Rightarrow z \in \mathrm{val}(u_\tau, F)$