IV

Fourth Lecture of

FORCING & THE CONTINUUM HYPOTHESIS

4 February 2025

RECAP

What we want to do

<u>Cohen Lemma</u>

For every finite $T \subseteq ZFC$ there is finite $T^* \subseteq ZFC$ s.t. if M ctm of T^*, then there is $N \supseteq M$ s.t. N ctm of $T + \neg CH$.

This reduces the problem to:

find ctm of T^* for sufficiently large finite $T^* \subseteq ZFC$.

Def. We call an assignment

$$\alpha \longmapsto Z_\alpha$$

a HIERARCHY, if

(i) Z_α is a trs set,
(ii) $\mathrm{Ord} \cap Z_\alpha = \alpha$,
(iii) $\alpha < \beta \Rightarrow Z_\alpha \subseteq Z_\beta$, and
(iv) λ limit $\rightarrow Z_\lambda = \bigcup_{\alpha < \lambda} Z_\alpha$.

If $\{Z_\alpha ; \alpha \in \mathrm{Ord}\}$ is a hierarchy, define $Z := \bigcup_{\alpha \in \mathrm{Ord}} Z_\alpha$. This … class … Ord

and $\rho_Z(x) := \min\{\alpha ; x \in Z_\alpha\}$ a notion of Z-rank.

What we need for this

<u>Lévy Reflection Theorem</u>

If Z is a hierarchy, φ is a formula. Then there unboundedly many ϑ s.t. φ is absolute between Z_ϑ and Z.

The technical tool to prove the LRT

Proposition 2.3.5 (Tarski–Vaught Test) *Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then, $\mathcal{M}$ is an elementary substructure if and only if, for any formula $\phi(v, \overline{w})$ and $\overline{a} \in M$, if there is $b \in N$ such that $\mathcal{N} \models \phi(b, \overline{a})$, then there is $c \in M$ such that $\mathcal{N} \models \phi(c, \overline{a})$.*

TVT

TVT_Φ : Let $M \subseteq N$ and Φ be a collection of formulas closed under subformulas. Then TFAE

(i) all formulas in Φ are absolute between M and N

(ii) for all $\varphi \in \Phi$, the <u>TV-condition</u> holds:
if $\varphi = \exists x \psi$, then f.a. $\vec{y} \in M$ if there is $a \in N$ s.t. $N \models \psi(a, \vec{y})$, then there is $b \in M$ s.t. $N \models \psi(b, \vec{y})$.

Lévy Reflection Theorem

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Warm-up

Let $(M, \in) \models \mathrm{ZFC}$. Find countable $N \subseteq M$ s.t. $(N, \in) \prec (M, \in)$.

Suppose $\vec{p} = (p_0, \ldots, p_n) \in M$ & $M \models \exists y \psi(y, \vec{p})$.
Let $w(\psi, \vec{p})$ be a witness for this:

$M \models \psi(w(\psi, \vec{p}), \vec{p})$ [if necessary, use AC]

[if $M \models \neg\exists y \psi(y, \vec{p})$, let $w(\psi, \vec{p}) = \emptyset$]

$N_0 := \emptyset$

$N_{i+1} := \{ w(\psi, \vec{p}) ; \psi \text{ formula } \& \vec{p} \in N_i^{<\omega} \}$

$N := \bigcup_{i \in \omega} N_i$.

(1) N is countable.

(2) $N \prec M$ by TVT.

<u>Remark</u>. In general, even if M is transitive, N is not.

E.g., if $\omega_1 \in M$, then

$$\exists x \, (x \text{ is the least uncountable ordinal}) \leftarrow \psi$$

is true in M

$$w(\psi, \emptyset) = \omega_1,$$

so $\omega_1 \in N$. But $\omega_1 \notin N$, since N is countable.

Relevant later!

Also, see ES#1.

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If $\{Z_\alpha ; \alpha \in \mathrm{Ord}\}$ is a hierarchy, we can define

$$Z := \bigcup_{\alpha \in \mathrm{Ord}} Z_\alpha.$$

This is a proper class with $\mathrm{Ord} \subseteq Z$.

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Proof of LRT. Fix φ and let Φ be its collection of subformulas. This is a finite set!

NTS: $\forall \alpha \exists \vartheta > \alpha$ s.t. $Z_\vartheta \models \varphi \Leftrightarrow Z \models \varphi$.

For each $\psi \in \Phi$ and $\vec{p} = (p_0, \ldots, p_n)$, write

$$o(\psi, \vec{p}) := \begin{cases} \text{least } \alpha \text{ s.t. } \exists y \in Z_\alpha \text{ with } Z \models \psi(y, \vec{p}) & \text{if it exists} \\ 0 & \text{o/w} \end{cases}$$

$$o(\vec{p}) := \max_{\psi \in \Phi} o(\psi, \vec{p})$$

$$\vartheta_0 := \alpha + 1$$

$$\vartheta_{i+1} := \sup\{o(\vec{p}) ; \vec{p} \in Z_{\vartheta_i}^{<\omega}\}$$

$$\vartheta := \sup_{i \in \omega} \vartheta_i$$

Then TVT implies that Z_ϑ and Z agree on φ. qed.

<u>Corollary</u> If $T \subseteq ZFC$ is finite, there is M transitive s.t. $M \models T$.

<u>Pf.</u> Let $\varphi := \bigwedge_{\psi \in T} \psi$. Since $ZFC \vdash \varphi$, we have that φ is true. By LRT, find ϑ s.t. $V_\vartheta \models \varphi$. V_ϑ is transitive.

q.e.d.

Remark about the proof of LRT:

Can you do the same if Φ is not finite?!?

Of course not: o/w we could prove

$\exists \vartheta \; V_\vartheta \models ZFC$

$\Longrightarrow \mathrm{Con}(ZFC)$

The problem is the case distinction in the definition of $o(\psi, \vec{p})$: it requires to check whether $\exists y \psi$ is true.

<u>Next goal</u> Obtain some $M \subseteq V_\vartheta$ countable s.t. $M \models \varphi$ and M is transitive.

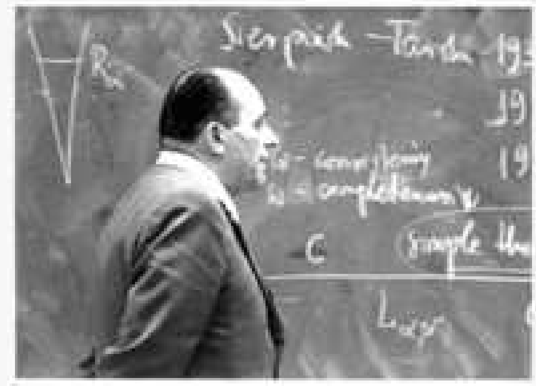
The analogue of 'subset collapse' is:

LEADER, Logic & Set Theory NOTES, §5

Theorem 4 (Mostowski's Collapsing Theorem). Let r be a relation on a set a that is well-founded and extensional. Then there exists a transitive set b, and a bijection $f : a \to b$ such that $(\forall x, y \in a)(x \; r \; y \Leftrightarrow f(x) \in f(y))$. Moreover, b and f are unique.

Remark. 'Well-founded' and 'extensional' are trivially necessary.

Andrzej Mostowski



Mostowski in 1973

Born	1 November 1913 Lemberg, Austria-Hungary
Died	22 August 1975 (aged 61) Vancouver, British Columbia, Canada
Nationality	Polish
Alma mater	University of Warsaw
Known for	Kleene–Mostowski hierarchy Mostowski collapse lemma Mostowski model

<u>Corollary</u>

For every $T \subseteq ZFC$ finite, there is a ctm of T.

<u>Proof.</u> Form $M \models T$ transitive by LRT.

[w.l.o.g., assume that T contains the axiom of extensionality.]

Use warm-up to obtain $N \prec M$ countable. This is extensional and well-founded, so by Mostowski find W transitive s.t.

$$(W, \in) \cong (N, \in).$$

Then $W \models T$ and $|W| = |N|$, so W is countable.

q.e.d.

Lectures $\overline{\text{V}}$ – $\overline{\text{VII}}$

Proof of Con(ZFC + CH) using Gödel's constructible universe.

Absoluteness is preserved under transfinite recursion

Let F, G, H be three operations.

RECURSION EQUATION ⊛

$$\left[\begin{array}{l} R(0,\vec{x}) := F(\vec{x}) \\ R(\alpha+1,\vec{x}) := G(\alpha, R(\alpha,\vec{x}), \vec{x}) \\ R(\lambda,\vec{x}) := H(\lambda, \{R(\alpha,\vec{x}) ; \alpha<\lambda\}, \vec{x}) \end{array}\right.$$

Note that for F, G, H there is a finite fragment $T_{F,G,H} \subseteq$ ZFC that proves the recursion theorem instance for F, G, H.

<u>Proof of recursion theorem:</u>

Attempts: set functions satisfying ⊛

L1: All attempts agree on their common domain.

L2: $\forall\alpha\exists r$ attempt $(\alpha,\vec{x}) \in \mathrm{dom}(r)$

$R(\alpha,\vec{x}) := y \;:\Longleftrightarrow\; \exists r$ attempt $(\alpha,\vec{x}) \in \mathrm{dom}(r)$ and $r(\alpha,\vec{x}) = y$

<u>Theorem</u> If $T \supseteq T_{F,G,H}$ and F, G, H are absolute for transitive models of T, then so is R defined by (⊛).

[Proof given: will be repeated in writing in Lecture $\overline{\text{V}}$]