III

Third Lecture of
FORCING & THE CONTINUUM HYPOTHESIS
1 February 2025

Transitive models

M transitive $\Longrightarrow (M, \in) \models$ Extensionality + Foundation

Δ_0, Σ_1, Π_1 formulas

Theorem: if M transitive, $T \subseteq ZFC$, $M \models T$, then

$$\varphi \begin{cases} \Delta_0^T \\ \Sigma_1^T \\ \Pi_1^T \end{cases} \Longrightarrow \begin{cases} \varphi \text{ absolute for } M \\ \varphi \text{ upwards absolute for } M \\ \varphi \text{ downwards absolute for } M \end{cases}$$

Operations absolute for M: requires both absoluteness of definition and closure

EXAMPLES — What is Δ_0?

1. $x \in y$
2. $x = y$
3. $x \subseteq y :\Leftrightarrow \forall w \in x (w \in y)$
4. $z = \{x\} \Leftrightarrow x \in z \wedge \forall w \in z (w = x)$
5. $z = \{x, y\}$
6. $z = (x, y) = \{\{x\}, \{x, y\}\}$
7. $z = \emptyset \Leftrightarrow \forall w \in z (w \neq w)$
8. $z = x \cup y \Leftrightarrow x \subseteq z \wedge y \subseteq z \wedge \forall w \in z (w$
9. $z = x \cap y$
10. $z = x \setminus y$
11. $z = x \cup \{x\}$
12. z is transitive
13. $z = \bigcup x$

More examples

14. z is an ordered pair
$\exists s \in z \exists d \in z \exists x \in s \exists y \in d$
$((\forall w \in s (w = x) \wedge \forall v \in d (v = x \vee v = y))$
$\wedge \forall w \in z (w = s \vee w = d))$
15. $z = a \times b$
16. z is a relation
17. $z = \mathrm{dom}(x)$
18. $z = \mathrm{ran}(x)$
19. z is a function
20. z is injective
21. z is surjective
22. z is bijective

ORDINALS

"x is an ordinal" $:\iff$ x is transitive $\wedge$
$(x, \in)$ is a wellorder.

We know: being wellfounded is not expressible in FOL

[ES#1]

Because all the models satisfy Foundation, we have that if M is the

$M \models x$ $\underbrace{\text{is transitive} \wedge (x,\in) \text{ is linearly ordered}}_{\text{clearly } \Delta_0}$

characterises ordinals.

So: being an ordinal is absolute for transitive models.

Thus $M \cap \mathrm{Ord} = \{x \in M;\ M \models x \text{ is an ordinal}\}$

This is transitive, thus there is $\alpha \in \mathrm{Ord}$ s.t.

$\alpha = M \cap \mathrm{Ord}$.

Also absolute:

"x is successor ordinal" $\quad \overset{\Delta_0}{\exists y \in x\, \forall w \in x}\ (w \in y \vee w = y)$

"x is limit ordinal"

"x is nonzero limit ordinal"

$x = \omega$ $\quad$ [$\forall w \in x$ w is a successor or 0 $\wedge$ x is$_{\text{nz}}$ limit]

ES#1: $x = \omega + \omega$
$x = \omega \cdot 3$
$x = \omega^2$ ...

CARDINALS

"x is a cardinal" $:\Longleftrightarrow$ x is an ordinal $\wedge$
$\forall f\ \underline{\forall y \in x}\ f: y \to x \Rightarrow$ f is not a surj.

?! not bounded ??

bdd

Observe: this is Π_1 and therefore downwards abs.

Remarks. 1. We may not want this to be absolute. If it does, we couldn't change cardinal behaviour.

2. We can't obviously bound $\forall f$ since the natural bound would be

$$\{h;\ h: y \to x\}$$
$$\mathcal{P}(y \times x)$$

These, however, are not (yet?!?) on our list of absolute concepts.

3. Note that neither 1. nor 2. is an argument since there could be an equivalent fmla that is Δ_0.

NON-ABSOLUTENESS

Assume that $M \models ZFC$ is transitive & countable.

$$M \cap \mathrm{Ord} = \alpha < \omega_1.$$

However, $M \models ZFC \implies M \models$ there are unctble cardinals.

Let $\beta < \alpha$ be s.t. $M \models \beta$ is the least unctble cardinal.

But β is a countable ordinal, so not a cardinal.

__Consequence__ All cardinal in M except for $\aleph_0$ are going to be fake

$\implies$ "x is a cardinal" can't be absolute.

__Note__: This also shows that "$x = \mathcal{P}(y)$" cannot be absolute.

Take y s.t. $M \models y = \mathcal{P}(\omega)$.

Then $y \subseteq \mathcal{P}(\omega)$, but is countable since $y \subseteq M$.

Thus $y \neq \mathcal{P}(\omega)$.

Therefore "$x = \mathcal{P}(y)$" is not absolute.

General proof strategy

Instead of substructures, we will restrict our attention to transitive substructures: in particular, to M transitive s.t. $M \models ZFC$

[$\forall x \; x \in M \Rightarrow x \subseteq M$
equivalently
$x \in M \wedge y \in x \Rightarrow y \in M$]

Cohen's proof becomes:

If M is a countable transitive set s.t. $M \models ZFC$, then there is ctble trs set $N \supseteq M$ s.t. $N \models ZFC + \neg CH$.

Q: Is this really solving the original problem? I.e., $Con(ZFC) \Longrightarrow Con(ZFC + \neg CH)$.

It's not obvious that $Con(ZFC) \Rightarrow$ there is a ctm

COUNTABLE TRANSITIVE MODEL

of ZFC!

A. That's not only not obvious, but false.

Let's prove that $Con(ZFC) \not\Rightarrow$ there is a ctm of ZFC.

Why? Note that $Con(ZFC)$, or $Con(T)$ for any T, is Δ_0. So, it's absolute for trs models.

So if M is a ctm of ZFC, then $Con(ZFC)$ is true, so by absoluteness $M \models Con(ZFC)$.

So $M \models ZFC + Con(ZFC)$. Contradicts Gödel's Incompleteness Thm.

We can get a proof of

$$\mathrm{Con}(ZFC) \Rightarrow \mathrm{Con}(ZFC + \neg CH)$$

via a trick (ES#1).

<u>Cohen Lemma</u>

For every finite $T \subseteq ZFC$ there is finite $T^* \subseteq ZFC$ s.t. if M ctm of T^*, then there is $N \supseteq M$ s.t. N ctm of $T + \neg CH$.

This reduces the problem to:

find ctm of T^* for sufficiently large finite $T^* \subseteq ZFC$.

For this:

Lévy Reflection Theorem

Def. We call an assignment
$$\alpha \longmapsto Z_\alpha$$
a HIERARCHY, if
(i) Z_α is a trs set,
(ii) $\mathrm{Ord} \cap Z_\alpha = \alpha$,
(iii) $\alpha < \beta \Longrightarrow Z_\alpha \subseteq Z_\beta$, and
(iv) λ limit $\longrightarrow Z_\lambda = \bigcup_{\alpha<\lambda} Z_\alpha$.

If $\{Z_\alpha ; \alpha \in \mathrm{Ord}\}$ is a hierarchy, we can define $Z := \bigcup_{\alpha \in \mathrm{Ord}} Z_\alpha$. This is a proper class with $\mathrm{Ord} \subseteq Z$.

and $\rho_Z(x) := \min\{\alpha ; x \in Z_\alpha\}$
a notion of Z-rank.

Paradigmatic example: von Neumann hierarchy V_α ; and V is the entire universe.

Lévy Reflection Theorem
If Z is a hierarchy, φ is a formula. Then there are unboundedly many ϑ s.t. φ is absolute between Z_ϑ and Z,

Proposition 2.3.5 (Tarski–Vaught Test) *Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then, $\mathcal{M}$ is an elementary substructure if and only if, for any formula $\phi(v, \overline{w})$ and $\overline{a} \in M$, if there is $b \in N$ such that $\mathcal{N} \models \phi(b, \overline{a})$, then there is $c \in M$ such that $\mathcal{N} \models \phi(c, \overline{a})$.*

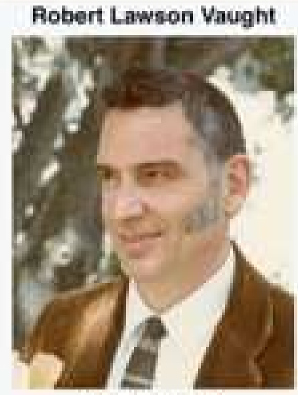
TVT

TVT_{Φ} : Let $M \subseteq N$ and Φ be a collection of formulas closed under subformulas. Then TFAE

(i) all formulas in Φ are absolute between M and N

(ii) for all $\varphi \in \Phi$, the TV-condition holds:
if $\varphi = \exists x \psi$, then f.a. $\vec{y} \in M$
if there is $a \in N$ s.t. $N \models \psi(a, \vec{y})$,
then there is $b \in M$ s.t. $N \models \psi(b, \vec{y})$.

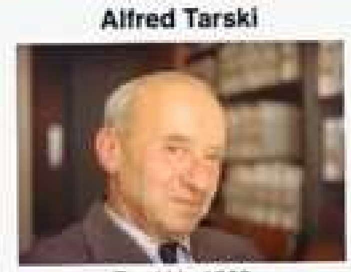
Robert Lawson Vaught



Vaught in 1974

Born	April 4, 1926 Alhambra, California
Died	April 2, 2002 (aged 75) Berkeley, California
Nationality	American
Alma mater	University of California, Berkeley

Alfred Tarski



Tarski in 1968

Born	Alfred Teitelbaum January 14, 1901 Warsaw, Congress Poland
Died	October 26, 1983 (aged 82) Berkeley, California, US
Nationality	Polish, American
Education	University of Warsaw (Ph.D., 1924)