Forcing & the Continuum Hypothesis

Lecture II

28 January 2025

Recap

Lecture I

Def. If $M \subseteq N$ and M, N are $\mathcal{L}$-structures and φ an $\mathcal{L}$-formula

We say φ is absolute between M & N if for all $x_1, \ldots, x_n \in M$

$$M \models \varphi(x_1, \ldots, x_n) \Longleftrightarrow N \models \varphi(x_1, \ldots, x_n)$$

$\Longrightarrow$ "upwards absolute"

$\Longleftarrow$ "downwards absolute"

Theorem (Substructure Lemma)

All atomic formulas are absolute between substructures.

Problem Since $\mathcal{L}_\in$ has no function & constant symbols, almost nothing is atomic!

Example

Lecture I $\longrightarrow$

A fundamental problem of non-absoluteness

$$\varphi_\emptyset(x) := \forall z\,(z \notin x)$$

"x is empty"

Consider $M \models \mathrm{ZFC}$. Therefore there are e, e' s.t. $M \models \varphi_\emptyset(e)$

$$M \models \forall z\,(z \in e' \Longleftrightarrow z = e).$$

Consider $N := M \setminus \{e\}$.

N is an $\mathcal{L}_\in$-substructure of M.

But $N \models \varphi_\emptyset(e')$

$M \models \neg\varphi_\emptyset(e')$

$\varphi_\emptyset$ is not absolute between substructures.

Note that $N := M \setminus \{e\}$ is not even a model of Extensionality $\longrightarrow$ ES#1

Def. If $M \subseteq N$ and M, N are $\mathcal{L}$-structures and φ an $\mathcal{L}$-formula

We say φ is absolute between M & N if for all $x_1, \ldots, x_n \in M$

$$M \models \varphi(x_1, \ldots, x_n) \iff N \models \varphi(x_1, \ldots, x_n)$$

$\Longrightarrow$ "upwards absolute"

$\Longleftarrow$ "downwards absolute"

Theorem (Substructure Lemma)

All atomic formulas are absolute between substructures.

φ is absolute for M if f.a. $x_1, \ldots, x_n \in M$

$M \models \varphi(x_1, \ldots, x_n) \iff \varphi(x_1, \ldots, x_n)$ is true

Clearly, if φ is absolute for M and absolute for N, then it's absolute between M and N.

Instead of substructures, we will restrict our attention to transitive substructures:

in particular, to M transitive s.t. $M \models ZFC$

[$\forall x \; x \in M \Rightarrow x \subseteq M$

equivalently

$x \in M \wedge y \in x \Rightarrow y \in M$]

Cohen's proof becomes:

If M is a countable transitive set s.t. $M \models ZFC$, then there is ctble trs set $N \supseteq M$ s.t.

$N \models ZFC + \neg CH$.

TRANSITIVE MODELS

Observation If M is transitive
$M \models$ "e is empty", then $e = \emptyset$.
[If $w \in e$, then $\left.\begin{array}{l} w \in e \\ e \in M \end{array}\right\} \Rightarrow w \in M$,
so $M \models w \in e \Rightarrow M \models$ "e is not empty".]

Lemma If M is transitive, then
$M \models$ Extensionality + Foundation.

Proof. Extensionality. $\forall x \forall y \underline{\forall w (w \in x \leftrightarrow w \in y)} \rightarrow x = y$

Take $x \neq y$, $x, y \in M$.
W.l.o.g., take $z \in x \setminus y$.
$\Rightarrow \left.\begin{array}{l} z \in x \\ x \in M \end{array}\right\} \Rightarrow z \in M$.

$M \models z \in x \wedge z \notin y$
$M \models \neg \forall w (w \in x \leftrightarrow w \in y)$.

Foundation. $\forall x (x \neq \emptyset \rightarrow \exists m (m \in x \wedge \forall w \neg (w \in m \wedge w \in x)))$

Take $x \in M$.
$M \models x \neq \emptyset \Rightarrow x$ is not empty
So find $\left.\begin{array}{l} m \in x \\ x \in M \end{array}\right\} \Rightarrow m \in M$ $\in$-minimal.

Therefore x has an $\in$-minimal elt in M. q.e.d.

ABSOLUTENESS for transitive models

We call a quantifier of the form

$$\exists x \in y \, \varphi$$
$$\forall x \in y \, \varphi$$

bounded quantifier

defined by

$$\exists x \in y \, \varphi := \exists x \, (x \in y \wedge \varphi)$$
$$\forall x \in y \, \varphi := \forall x \, (x \in y \rightarrow \varphi)$$

A class of formulas Γ is closed under bounded quantification if whenever φ is in Γ, then so are $\exists x \in y \, \varphi$ and $\forall x \in y \, \varphi$.

Def. Δ_0 is the smallest class of formulas containing the atomic formulas that is closed under propositional connectives and bounded quantification.

Let T be any theory, then Δ_0^T is the class of formulas equivalent to a Δ_0-formula in T.

Theorem Δ_0 formulas are absolute for transitive models.

Proof. By induction:

1. All atomic fmlas are absolute by substructure lemma.
2. Propositional connectives exactly the same proof as in substructure lemma.
3. Assume that φ is absolute and show that $\exists x \in y\, \varphi$ and $\forall x \in y\, \varphi$ are absolute.

$\exists x \in y\, \varphi$. If $\exists x \in y\, \varphi$ is true for some $y \in M$.

Then $\left.\begin{matrix} x \in y \\ y \in M \end{matrix}\right\} \Rightarrow x \in M$

By induction hypothesis $M \models \varphi(x,y)$

Thus $M \models \exists x\, (x \in y \wedge \varphi(x,y))$.

If $M \models \exists x\, (x \in y \wedge \varphi(x,y))$

then $x \in y \wedge \varphi(x,y)$ is true.

q.e.d.

Corollary If $T \subseteq ZFC$ is any theory and $M \models T$ is transitive, then Δ_0^T-formulas are absolute for M.

Def. A formula is called Σ_1 if it is of the form $\exists x_1 \ldots \exists x_n \varphi$ where φ is Δ_0;
it is called Π_1 if it is of the form $\forall x_1 \ldots \forall x_n \varphi$ where is Δ_0.
[Same for Σ_1^T, Π_1^T.]

Corollary Σ_1^T-formulas are upwards absolute for T's model
Π_1^T downwards

Proof Just definition of the semantics of $\exists, \forall$.
q.e.d.

EXAMPLES What is Δ_0?

1. $x \in y$
2. $x = y$
3. $x \subseteq y :\Leftrightarrow \forall w \in x (w \in y)$
4. $z = \{x\} :\Leftrightarrow x \in z \wedge \forall w \in z (w = x)$
5. $z = \{x, y\}$
6. $z = (x, y) = \{\{x\}, \{x, y\}\}$
7. $z = \emptyset :\Leftrightarrow \forall w \in z (w \neq w)$
8. $z = x \cup y \Leftrightarrow x \subseteq z \wedge y \subseteq z \wedge \forall w \in z (w \in x \vee w \in y)$
9. $z = x \cap y$
10. $z = x \setminus y$
11. $z = x \cup \{x\}$
12. z is transitive
13. $z = \bigcup x$

Def. If M transitive set

$$F : M^n \longrightarrow M$$

note that this means that M is closed under F

We say F is absolute for M if there is a formula Φ absolute for M s.t..

$$F(x_1, \dots, x_n) = z \iff \Phi(x_1, \dots, x_n, z).$$

Observation So, if M is closed under pairing ($\forall x, y \in M \ \{x,y\} \in M$), then the pairing operation $x, y \longmapsto \{x,y\}$ is absolute, and therefore

$$M \models \text{Pairing}.$$

Similarly for union

Lemma If φ is absolute for M
$F, G_1, \dots, G_n$ are absolute oper. on M
then

$$\psi(x_1, \dots, x_m) := \varphi(G_1(x_1, \dots, x_m), \dots, G_n(x_1, \dots, x_m))$$

$$H(x_1, \dots, x_m) := F(G_1(x_1, \dots, x_m), \dots, G_n(x_1, \dots, x_m))$$

are absolute for M.

Check the definitions!

More examples

14. z is an ordered pair

$\exists s \in z \exists d \in z \exists x \in s \exists y \in d$
$((\forall w \in s (w = x) \wedge \forall v \in d (v = x \vee v = y))$
$\wedge \forall w \in z (w = s \vee w = d))$

15. $z = a \times b$

16. z is a relation

17. $z = dom(x)$

18. $z = ran(x)$

19. z is a function

20. z is injective

21. z is surjective

22. z is bijection