Forcing & the Continuum Hypothesis (L16)

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The method of forcing is one of the most important model constructions in set theory and its versatility is the reason for the plethora of independence results in set theory. It was developed to solve one of the most important foundational problems of the 20th century: determining the cardinality of the set of real numbers. Cantor's *Continuum Hypothesis* asserts that the cardinality of the set of real numbers has the smallest possible value:

> Every infinite set of reals is either equinumerous with the set of natural numbers or equinumerous with the set of all real numbers. Equivalently, $2^{\aleph_0} = \aleph_1$. (CH)

When he presented his list of twenty-three problems for the 20th century at the *International Congress of Mathematicians* in Paris in 1900, David Hilbert listed the the question whether the Continuum Hypothesis is true as the very first problem on his list. It turned out that this question cannot be solved on the basis of ZFC: in 1938, Kurt Gödel showed that CH cannot be disproved in ZFC; in 1963, Paul Cohen invented the method of forcing to show that CH cannot be proved in ZFC.

In this course we shall study the basics of the method of forcing in order to present Cohen's proof. The course will discuss:

1. models of set theory (in particular, transitive models, the Lévy reflection theorem, and the theory of absoluteness);
2. the construction of the generic extension of a countable transitive model (in particular, the syntactic and semantic forcing relation, the forcing theorem, and the generic model theorem);
3. showing relative consistency proofs with the method of forcing;
4. applications to cardinal arithmetic (in particular, preservation results and the consistency of $\neg$CH and related results).

David Hilbert

Hilbert in 1912

Born	23 January 1862 Königsberg or Wehlau, Kingdom of Prussia
Died	14 February 1943 (aged 81) Göttingen, Nazi Germany
Education	University of Königsberg (PhD)

Kurt Gödel

Gödel c. 1926

Born	Kurt Friedrich Gödel April 28, 1906 Brünn, Austria-Hungary (now Brno, Czech Republic)
Died	January 14, 1978 (aged 71) Princeton, New Jersey, U.S.

1938: GÖDEL

ZFC+CH is consistent

[Con(ZFC) $\Rightarrow$ Con(ZFC+CH)].

1962: COHEN

Con(ZFC) $\Rightarrow$ Con(ZFC+ $\neg$CH)

Paul Cohen

American mathematician

Paul J. Cohen

Born	April 2, 1934 Long Branch, New Jersey, U.S.
Died	March 23, 2007 (aged 72) Stanford, California, U.S.
Alma mater	University of Chicago (MS, PhD)
Known for	Cohen forcing Continuum hypothesis
Awards	Bôcher Prize (1964) Fields Medal (1966) National Medal of Science (1967)

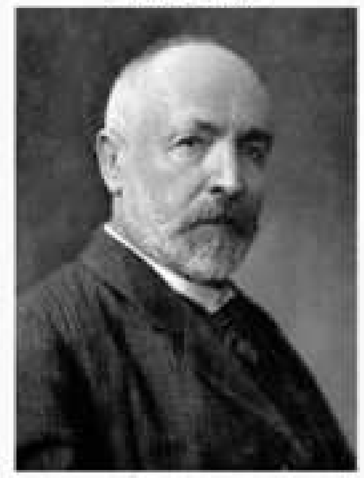
THE CONTINUUM HYPOTHESIS

Friedrich Moritz Hartogs

Friedrich Hartogs

Born	20 May 1874 Brussels
Died	18 August 1943 (aged 69) Munich

Georg Cantor



Cantor, c.1910

Born	Georg Ferdinand Ludwig Philipp Cantor 3 March 1845 Saint Petersburg, Russian Empire
Died	6 January 1918 (aged 72) Halle, Province of Saxony, German Empire
Nationality	German

Hartogs's Thm

For every X, there is (least) ordinal α s.t. there is no inj. from α to X.

The Hartogs aleph of X

$\aleph(X)$

$$X \longmapsto \aleph(X)$$

Cantor's Theorem

For every X, there is no inj. from $\mathcal{P}(X)$ into X.

$$2^{|X|}$$

$$X \longmapsto \mathcal{P}(X)$$

Using AC, wellorder $\mathcal{P}(X)$ and get ordinal $2^{|X|}$.

$$\boxed{\aleph(X) \leq 2^{|X|}.}$$

$$\aleph_0 := \mathbb{N}$$
$$\aleph_{\alpha+1} := \aleph(\aleph_\alpha)$$
$$\aleph_\lambda := \bigcup_{\alpha<\lambda} \aleph_\alpha$$

$$\beth_0 := \mathbb{N}$$
$$\beth_{\alpha+1} := 2^{\beth_\alpha}$$
$$\beth_\lambda := \bigcup_{\alpha<\lambda} \beth_\alpha$$

Clearly, $\aleph_\alpha \leq \beth_\alpha$.

CH: $\aleph_1 = \beth_1$.

GCH: $\forall\alpha \ \aleph_\alpha = \beth_\alpha$.

Why "continuum"?

__Lemma__ CH $\iff$ $\forall X \subseteq \mathbb{R}$ X is uncountable $\implies X \sim \mathbb{R}$

[there is a bijection]

__Pf.__ Clearly $\beth_1 = |\mathbb{R}|$.

Therefore "$\implies$" is obvious.

For "$\Leftarrow$", suppose $|\mathbb{R}| > \aleph_1$.

Wellorder the reals in order type $\kappa > \aleph_1$:

$$\{ r_\alpha ; \alpha < \kappa \}$$

Consider $X := \{ r_\alpha ; \alpha < \omega_1 \} \subseteq \mathbb{R}$

That's a subset of the reals of card $\aleph_1$ not in bij. with $\mathbb{R}$. q.e.d.

GÖDEL:

$\mathrm{Con}(ZFC) \Rightarrow \mathrm{Con}(ZFC + CH)$

COHEN:

$\mathrm{Con}(ZFC) \Rightarrow \mathrm{Con}(ZFC + \neg CH)$

Relative consistency proofs.

By Completeness Theorem, this means:

If there is $M \models ZFC$, then I can construct $N \models ZFC + (\neg) CH$

ANALOGY from algebra.

$\mathcal{L}_{\text{Fields}} = \{0, 1, +, \cdot, -, ^{-1}\}$

Axioms of fields: Fields

$\varphi_{\sqrt{2}} := \exists x \, (x \cdot x = 1 + 1)$ $\quad \mathbb{Q} \models \neg \varphi_{\sqrt{2}}$.

<u>Idea</u> Start with $\mathbb{Q}$ and extend $\mathbb{Q}$ to get $F \models \text{Fields} + \varphi_{\sqrt{2}}$.

Construct by ALGEBRAIC CLOSURE

Obtain $\mathbb{Q}(\sqrt{2}) \models \text{Fields} + \varphi_{\sqrt{2}}$.

This is easy because everything that matters (Fields and $\varphi_{\sqrt{2}}$) is determined by equations; all formulas we need to check are atomic.

<u>Def.</u> If $M \subseteq N$ and M, N are $\mathcal{L}$-structures and φ an $\mathcal{L}$-formula

We say <u>φ is absolute between M & N</u> if for all $x_1, \dots, x_n \in M$

$$M \models \varphi(x_1, \dots, x_n) \iff N \models \varphi(x_1, \dots x_n)$$

$\Longrightarrow$ "upwards absolute"

$\Longleftarrow$ "downwards absolute".

<u>Theorem</u> (Substructure Lemma)

All atomic formulas are absolute between substructures.

What if we have models of ZFC?

$\mathcal{L}_\in = \{\in\}$. No function symbols nor constant symbols.

So: almost nothing is atomic.

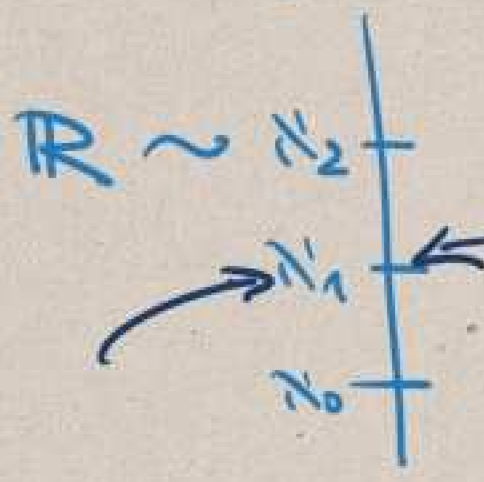
$M \subseteq N \iff M$ is an $\mathcal{L}_\in$-substructure of N

And: the formulas we care about are definitely not atomic and very complex.

Try to imagine a proof of:

If $M \models ZFC$ then there is $N \supseteq M$
$N \models ZFC + CH$.

w.l.o.g. $M \models \neg CH$, for simplicity say $\beth_1 = \aleph_2$.



What gets rid of $\aleph_1$?

E.g., a surjection $f : \mathbb{N} \longrightarrow \aleph_1$.

Maybe we can force $M[f] \supseteq M$ to get a smallest model
$M[f] \models ZFC$.

Clearly, in $M[f]$, $\aleph_1^M$ is not a cardinal anymore.

Does that show CH?

All sorts of things could happen:

(1) $\mathbb{R}^{M[f]} \neq \mathbb{R}^M$

(2) $\aleph_2^M$ is not a cardinal either

A fundamental problem of non-absoluteness

$$\varphi_\emptyset(x) := \forall z\,(z \notin x)$$

"x is empty"

Consider $M \models ZFC$. Therefore there are e, e' s.t. $M \models \varphi_\emptyset(e)$

$$M \models \forall z\,(z \in e' \iff z = e).$$

Consider $N := M \setminus \{e\}$.

N is an $\mathcal{L}_\in$-substructure of M.

But $N \models \varphi_\emptyset(e')$

$M \models \neg\varphi_\emptyset(e')$

$\varphi_\emptyset$ is not absolute between substructures.

Instead of substructures, we will restrict our attention to transitive substructures:

in particular, to M transitive s.t. $M \models ZFC$

$[\forall x\ \ x \in M \Rightarrow x \subseteq M$

equivalently

$x \in M \wedge y \in x \Rightarrow y \in M]$