Forcing & the Continuum Hypothesis
Lent Term 2025
Part III of the Mathematical Tripos
University of Cambridge
Prof. Dr. B. Löwe, L. Gardiner

Example Sheet #3

Examples Classes.

Examples Class #1. Thursday 13 February 2025, 1:30–3:30pm, **MR3**.
Examples Class #2. Thursday 27 February 2025, 1:30–3:30pm, **MR3**.
Examples Class #3. Thursday 20 March 2025, 1:30–3:30pm, **MR3**.

Presentation. Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

Marking. You can submit all of your work to Lyra Gardiner (`lag44`) as a *single pdf file* by e-mail or hand it in on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission.

Unless otherwise specified, M is a countable transitive model of set theory, $(\mathbb{P}, \leq, \mathbf{1}) \in M$ is a partial order, and G is $\mathbb{P}$-generic over M.

(28) We call $p \in \mathbb{P}$ an *atom* if there are no $q, r \leq p$ such that $q \perp r$. We call $\mathbb{P}$ *non-atomic* (sometimes called *splitting*) if it has no atoms. Prove that $\mathbb{P}$ is non-atomic if and only if for all G that are $\mathbb{P}$-generic over M, we have $G \notin M$.

(29) Presentation Example. Let M be a countable transitive model of set theory and G be $\mathbb{P}$-generic over M. Complete the proof of the Generic Model Theorem given in the lectures by proving that $M[G]$ is a model of the powerset axiom. [You may use the Forcing Theorem without proof.]

(30) Let $E \subseteq \mathbb{P}$, $E \in M$, and $p \in \mathbb{P}$. Prove the following.

 (i) Either $G \cap E \neq \varnothing$ or there is $q \in G$ such that for all $r \in E$, we have $r \perp q$.
 (ii) If $p \in G$ and E is dense below p, then $G \cap E \neq \varnothing$.

(31) Complete the proof of the Forcing Theorem given in the lectures by proving the induction steps for $\wedge$ and $\exists$.

(32) Let M be a countable transitive model of set theory and $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a non-atomic partial order (cf. (28)). Define recursively $M_0 := M$ and $M_{i+1} := M_i[G_i]$ where G_i is $\mathbb{P}$-generic over M_i. Prove that $\bigcup_{i \in \mathbb{N}} M_i$ is not a model of ZFC.

(33) A family of finite sets $\mathcal{D}$ is called a *Δ-system* if there is a finite set R (called the *root of the Δ-system*) such that for all $D, D' \in \mathcal{D}$, if $D \neq D'$, then $D \cap D' = R$. Show that any uncountable family of finite sets contains an uncountable Δ-system.

(*Hint.* Argue that you can assume w.l.o.g. that all elements of the family have the same size and prove the claim by induction on the size of the elements of the family.)

(34) If $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}})$ are partial orders, then a function $i\colon \mathbb{P} \to \mathbb{Q}$ is called a *complete embedding* if

(a) i is order preserving, i.e., if $p \leq_{\mathbb{P}} p'$, then $i(p) \leq_{\mathbb{Q}} i(p')$;

(b) i preserves incompatibility in both directions, i.e., $p \perp_{\mathbb{P}} p'$ if and only if $i(p) \perp_{\mathbb{Q}} i(p')$; and

(c) for all $q \in \mathbb{Q}$ there is a $p \in \mathbb{P}$ such that for all $p' \leq_{\mathbb{P}} p$, we have that $i(p')$ and q are compatible in $\mathbb{Q}$.

Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding with $i, \mathbb{P}, \mathbb{Q} \in M$ and let H be $\mathbb{Q}$-generic over M. Show that $G := \{p \in \mathbb{P}\,;\, i(p) \in H\}$ is $\mathbb{P}$-generic over M and that $M[G] \subseteq M[H]$.

(35) If $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}})$ are partial orders, then a function $i : \mathbb{P} \to \mathbb{Q}$ is called a *dense embedding* if

(a) i is order preserving, i.e., if $p \leq_{\mathbb{P}} p'$, then $i(p) \leq_{\mathbb{Q}} i(p')$;

(b) i preserves incompatibility, i.e., if $p \perp_{\mathbb{P}} p'$, then $i(p) \perp_{\mathbb{Q}} i(p')$; and

(c) the image of $\mathbb{P}$ under i is dense in $\mathbb{Q}$.

Show that every dense embedding is a complete embedding.

(36) Let $\mathbb{P} := \mathrm{Fn}(\omega, \omega)$ and $\mathbb{Q}$ be any countable non-atomic partial order (cf. (28)). Show that there is a dense embedding from $\mathbb{P}$ to $\mathbb{Q}$.

(37) We say that $\mathbb{P}$ *preserves cofinalities* if for every $\mathbb{P}$-generic filter G over M and every limit ordinal $\lambda \in M$, we have that $\mathrm{cf}(\lambda)^M = \mathrm{cf}(\lambda)^{M[G]}$. Prove that if $\mathbb{P}$ preserves cofinalities, then it preserves cardinals.

(38) If κ is a cardinal, we say that $\mathbb{P}$ has the κ-c.c. if every antichain in $\mathbb{P}$ has cardinality smaller than κ. (Thus, the c.c.c. is the $\aleph_1$-c.c.) If κ is a cardinal in M, we say that $\mathbb{P}$ *preserves cardinals* $\geq \kappa$ if for every $\mathbb{P}$-generic filter G over M and every $\lambda \geq \kappa$, we have that $M \models$"λ is a cardinal" if and only if $M[G] \models$"λ is a cardinal". Show that if $M \models$"κ is a regular cardinal" and $M \models$"$\mathbb{P}$ has the κ-c.c.", then $\mathbb{P}$ preserves cardinals $\geq \kappa$.

(39) Let $f, g\colon \omega \to \omega$ be functions. We say that g *bounds* f if for all but finitely many n, we have that $f(n) \leq g(n)$. Let $\mathbb{C} := \mathrm{Fn}(\omega, \omega)$ and G be $\mathbb{C}$-generic over M. Show that there is an $f \in M[G]$ that is not bounded by any function in M.

(40) If $f, g\colon \omega \to \omega$, we say that f and g are *infinitely often equal* if $\{n\,;\, f(n) = g(n)\}$ is infinite. Let $\mathbb{C} := \mathrm{Fn}(\omega, \omega)$ and G be $\mathbb{C}$-generic over M. Show that there is an $f \in M[G]$ such that for all functions $g \in M$, f and g are infinitely often equal.

(41) Presentation Example. If $f, g\colon \omega \to \omega$, we say that f and g are *eventually different* if there is some n such that for all $k \geq n$, we have that $f(k) \neq g(k)$.

We write $\omega^{<\omega}$ for the set of finite sequences of natural numbers, ω^ω for the set of functions from ω to ω, and $[\omega^\omega]^{<\omega}$ for the set of finite subsets of ω^ω. Let $\mathbb{E} := \{(s, S)\,;\, s \in \omega^{<\omega}, S \in [\omega^\omega]^{<\omega}\}$ ordered by $(s, S) \leq (t, T)$ if and only if $t \subseteq s$, $T \subseteq S$, and for all $n \in \mathrm{dom}(s)\backslash\mathrm{dom}(t)$ and all $g \in T$, we have that $s(n) \neq g(n)$.

Let G be $\mathbb{E}$-generic over M. Prove that there is a function $f \in M[G]$ such that for all functions $g \in M$, f and g are eventually different.

(42) Consider the partial orders $\mathbb{P}_i := \mathrm{Fn}(\omega, \aleph_i^M)$ and assume that G_i is $\mathbb{P}_i$-generic over M. Let N be any countable transitive model such that $\{G_i\,;\, i \in \omega\} \subseteq N$. Show that there is an M-cardinal $\aleph_\alpha^M$ for $\alpha \geq \omega$ that is countable in N.