Forcing & the Continuum Hypothesis
Lent Term 2025
Part III of the Mathematical Tripos
University of Cambridge
Prof. Dr. B. Löwe, L. Gardiner

Example Sheet #2

Examples Classes.

Examples Class #1. Thursday 13 February 2025, 1:30–3:30pm, **MR3**.
Examples Class #2. Thursday 27 February 2025, 1:30–3:30pm, **MR3**.
Examples Class #3. Thursday 20 March 2025, 1:30–3:30pm, **MR3**.

Presentation. Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

Marking. You can submit all of your work to Lyra Gardiner (`lag44`) as a *single pdf file* by e-mail or hand it in on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission.

Corrections. A number of corrections were made from an earlier version. For (18) & (21), please use a slightly more liberal notion of *hierarchy* than used in the lectures: $\alpha \mapsto Z_\alpha$ is a *hierarchy* if (a) all Z_α are transitive, (b) for all α, $\alpha \subseteq Z_\alpha$, (c) if $\alpha < \beta$, then $Z_\alpha \subseteq Z_\beta$, and (d) if λ is a limit, then $\bigcup_{\xi<\lambda} Z_\xi = Z_\lambda$. Note what goes wrong in (18) & (21) with the stronger condition $\mathrm{Ord} \cap Z_\alpha = \alpha$ from the lectures. Also, (16) was corrected to say $\mathcal{D}(Z_\alpha)$ instead of $\mathbf{L}_\alpha$.

(15) An ordinal γ is called a *gamma number* if for all $\alpha, \beta < \gamma$, we have $\alpha + \beta < \gamma$.

- (i) Show that $\gamma > 0$ is a gamma number if and only if there is an ordinal ξ such that $\gamma = \omega^\xi$.
- (ii) Show that if M is a countable transitive model of ZFC and $\gamma := \mathrm{Ord} \cap M$, then γ is a gamma number.
- (iii) Show that there is some $E \in \mathbf{L}_{\omega+1}$ such that (ω, E) is isomorphic to a gamma number $\gamma > \omega$.

(16) Let $\{Z_\alpha\,;\, \alpha \in \mathrm{Ord}\}$ be a definable hierarchy in the sense of Lecture III. Assume that for all α there is a γ such that $\mathcal{D}(Z_\alpha) \subseteq Z_\gamma$. Show that $Z := \bigcup_{\alpha \in \mathrm{Ord}} Z_\alpha$ is a model of the powerset axiom.

(17) Show that the axiom schema of replacement holds in $\mathbf{L}$.

(18) Assume that M is a countable transitive model of ZFC with $\mathrm{Ord} \cap M = \gamma < \omega_1$. Let $\{\gamma_n\,;\, n \in \omega\}$ be an increasing sequence such that $\bigcup_{n\in\omega} \gamma_n = \gamma$, $a_n := \{0, 2, 4, ..., 2n\}$ the set of the first $n+1$ many even numbers, and $A_\alpha := \{a_n\,;\, \gamma_n < \alpha\}$. Define $Z_\alpha := \alpha \cup A_\alpha$. Prove the following statements.

- (i) The set $\{Z_\alpha\,;\, \alpha \in \gamma\}$ is a hierarchy in M.
- (ii) The set $Z := \bigcup_{\alpha\in\gamma} Z_\alpha$ is not a model of the powerset axiom.

(19) Using the idea of Example (18) and assuming that there are infinitely many non-constructible subsets of ω, argue that the word "definable" in the statement of (16) is necessary: under the assumption, the statement of (16) does not hold for non-definable hierarchies. Where in the proof of (16) did you use that the hierarchy was definable? Why is it not possible to give an example without making some assumption about the existence of non-constructible sets?

(20) For an ordinal α, we say that *a constructible subset of ω is constructed at α* if

$$\mathbf{L}_{\alpha+1} \backslash \mathbf{L}_\alpha \cap \wp(\omega) \neq \varnothing.$$

Show that

(i) a constructible subset of ω is constructed at ω;

(ii) if $\alpha \geq \omega_1$, no constructible subset of ω is constructed at α; and

(iii) that there are unboundedly many $\alpha < \omega_1$ where a constructible subset of ω is constructed.

(21) Let x be any transitive set and define by transfinite recursion:

$$\begin{aligned} \mathbf{L}_0(x) &:= x, \\ \mathbf{L}_{\alpha+1}(x) &:= \mathcal{D}(\mathbf{L}_\alpha(x)), \\ \mathbf{L}_\lambda(x) &:= \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha(x) \text{ (for } \lambda \text{ limit).} \end{aligned}$$

Show that

(i) $\{\mathbf{L}_\alpha(x)\,;\, \alpha \in \mathrm{Ord}\}$ is a hierarchy;

(ii) if x is countable and $\alpha \geq \omega$, then $|\mathbf{L}_\alpha(x)| = |\alpha|$; and

(iii) if x is countable, then $\mathbf{L}(x) \models \mathsf{GCH}$.

(22) Presentation Example. Using the models from (21), let α_x be the ordinal such that $\mathbf{L}(x) \models$"α_x is the least uncountable cardinal". Show that if for all $x \subseteq \omega$, we have that $\alpha_x < \aleph_1$, then $\aleph_1$ is a regular limit cardinal in $\mathbf{L}$.

(23) Let $(\mathbb{P}, \leq)$ be a partial order and $p \in \mathbb{P}$. Show that

(a) if D is dense below p and $r \leq p$, then D is dense below r;

(b) if $\{r\,;\, D \text{ is dense below } r\}$ is dense below p, then D is dense below p.

(24) Presentation Example. We say that G is $\mathbb{P}$-*antichain generic over* M if for every maximal $\mathbb{P}$-antichain $A \in M$, we have $A \cap G \neq \varnothing$. We call a set B a $\mathbb{P}$-*bar* if for every $p \in \mathbb{P}$ there is a $b \in B$ such that p and b are compatible. We say that G is $\mathbb{P}$-*bar generic over* M if for every $\mathbb{P}$-bar $B \in M$ we have that $B \cap G \neq \varnothing$.

Let $\mathbb{P} \in M$, and G be a filter over $\mathbb{P}$. Show that the following are equivalent:

(i) G is $\mathbb{P}$-generic over M,

(ii) G is $\mathbb{P}$-antichain generic over M, and

(iii) G is $\mathbb{P}$-bar generic over M.

(25) Let M be a transitive model of set theory and $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a partial order. Suppose $\sigma, \tau \in \mathrm{Names}(M, \mathbb{P})$ and that D is a filter on $\mathbb{P}$. Show that $\mathrm{val}(\sigma \cup \tau, D) = \mathrm{val}(\sigma, D) \cup \mathrm{val}(\tau, D)$.

(26) Let M be a transitive model of set theory, $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a partial order, and $x \in M$. Recursively define $\mathrm{can}(x) := \{(\mathrm{can}(y), p)\,;\, y \in x \wedge p \in \mathbb{P}\}$. Show that if D is a filter, then $\mathrm{val}(\mathrm{can}(x), D) = x$.

(27) Let M be a transitive model of set theory, $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a partial order, and D is a filter on $\mathbb{P}$ with $D \neq \mathbb{P}$. Show that there is a proper class of names τ such that $\mathrm{val}(\tau, D) = \varnothing$.