Forcing & the Continuum Hypothesis
Lent Term 2025
Part III of the Mathematical Tripos
University of Cambridge
Prof. Dr. B. Löwe, L. Gardiner

Example Sheet #1

Examples Classes.

Examples Class #1. Thursday 13 February 2025, 1:30–3:30pm, **MR3**.
Examples Class #2. Thursday 27 February 2025, 1:30–3:30pm, **MR3**.
Examples Class #3. Thursday 20 March 2025, 1:30–3:30pm, **MR3**.

Presentation. Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

Marking. You can submit all of your work to Lyra Gardiner (`lag44`) as a *single pdf file* by e-mail or hand it to them on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission.

(1) Let M be a set such that $(M, \in) \models \mathsf{ZFC}$ and $e \in M$ such that $(M, \in) \models$"e is empty". Consider $N := M \backslash \{e\}$ as in Lecture I and check which of the axioms of ZFC hold in $(N, \in)$.

(2) Show that the following properties can be expressed by Δ_0-formulas.

  (a) $z = y \times y$;
  (b) z is a function;
  (c) z is a group;
  (d) z is a linear order;
  (e) z is a set with exactly two elements.

(3) Consider the structure $(\omega, \in)$ and determine which of the axioms of ZFC hold in it. Show that it is not closed under ordered pairs.

(4) Consider the closure of ω under pairing: $M_0 := \omega$, $M_{i+1} := M_i \cup \{\{x, y\}\,;\, x, y \in M_i\}$, and $M_\infty := \bigcup_{i\in\omega} M_i$. Show that M_∞ is a transitive set and determine which of the axioms of ZFC hold in it. Identify which functions f with $\mathrm{dom}(f) \subseteq \omega$ and $\mathrm{ran}(f) \subseteq \omega$ are in M_∞.

(5) Presentation Example. Show that if ZFC is consistent, then there is a model $(M, E) \models \mathsf{ZFC}$ such that E is an ill-founded relation. Can you build such an M such that (N, E) is ill-founded where $N := \{x \in M\,;\,(M, E) \models$"$x$ is a natural number"$\}$?

(6) Show that the following properties can be expressed by Δ_0-formulas.

(a) $z = \omega \cdot 3$;

(b) $z = \omega^2$;

(c) $z = \omega^2 + \omega$.

(7) Presentation Example. Let M be a transitive set model of ZFC and let $\alpha \in M$ be such that $(M, \in) \models$"α is the least uncountable cardinal". Show that if α is countable, then there is some $A \subseteq \omega$ such that $A \notin M$.

(8) The *Mirimanoff rank* is defined by $\varrho(x) := \min\{\alpha\,;\,x \in \mathbf{V}_{\alpha+1}\}$. Show that for any ordinal α and any sets x and y, the following hold:

(a) if $x \in y$, then $\varrho(x) < \varrho(y)$;

(b) $\varrho(x) = \sup\{\varrho(y) + 1\,;\,y \in x\}$;

(c) $\varrho(\alpha) = \alpha$.

(9) Let $\vartheta \geq \omega + 2$ be an ordinal and assume that $M \subseteq \mathbf{V}_\vartheta$ is a countable elementary submodel constructed using the Tarski-Vaught test, i.e., by iteratively collecting witnesses to all existential formulas true in $\mathbf{V}_\vartheta$ (cf. Lecture IV). Show that M is not transitive. What happens in the cases $\vartheta = \omega$ and $\vartheta = \omega + 1$?

(10) Assume that for every finite $T \subseteq \mathsf{ZFC}$ there is some finite $T^* \subseteq \mathsf{ZFC}$ such that if M is a countable transitive model of T^*, then there is a countable transitive model $N \supseteq M$ of $T + \varphi$. Show that if ZFC is consistent, then so is $\mathsf{ZFC} + \varphi$.

[*Hint.* We proved in the lectures (using the Lévy Reflection Theorem and Löwenheim-Skolem) that ZFC proves that for every finite $T^* \subseteq \mathsf{ZFC}$, there is a countable transitive model of T^*.]

(11) Let $\Sigma := \{\in, =, \wedge, \vee, \neg, \exists, \forall, (,)\}$ and V be a countable set of variables disjoint from Σ. Fix some bijection $c\colon \Sigma \cup V \to \omega$. Give the recursive definition that specifies which finite ω-sequences are (codes for) well-formed formulas and argue that this set is absolute for transitive models of some finite $T \subseteq \mathsf{ZFC}$. Discuss what T is.

(12) Fix a set X and $E \subseteq X \times X$. Using your encoding from (11), provide a recursive definition of "$(X, E) \models \varphi$" that is absolute for transitive models of some finite $T \subseteq \mathsf{ZFC}$ containing X and E. Discuss what T is. Highlight why it is important that X is an element of the model. Use this to argue that $D(\varphi, p, X)$ and $\mathcal{D}(X)$, as defined in Lecture V, are absolute operations for transitive models of some finite T^*. Again, discuss what T^* is.

(13) Show that $\alpha \mapsto \mathbf{L}_\alpha$ is a hierarchy in the sense of Lecture III.

(14) We call λ a *beth fixed point* if $\lambda = \beth_\lambda$. Show that for $\lambda > \omega$, we have that $|\mathbf{V}_\lambda| = |\mathbf{L}_\lambda|$ if and only if λ is a beth fixed point.