Large Cardinals
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Model Solutions for Example Sheet #3

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(33) Give concrete functions $f : \kappa \to \kappa$ such that $(f) = (\mathrm{id}) + 1$, $(f) = (\mathrm{id}) + \omega_1$, $(f) = (\mathrm{id}) \cdot 2$, $(f) = (\mathrm{id})^+$, and $(f) = (\mathrm{id})^{++}$. Fix $\xi < \kappa$ and consider the function $f(\alpha) := \xi$ if α is even and $f(\alpha) := \alpha$ if α is odd. What can we say about the relation between (id) and (f)?

[*Remark.* As usual, an ordinal α is even if it is of the form $\lambda + 2n$ where λ is a limit ordinal and n is a natural number.]

Solution. Let $f_1 : \kappa \to \kappa$ be given by $\alpha \mapsto \alpha + 1$. Clearly (f_1) is an ordinal, and furthermore $\{\alpha < \kappa : f(\alpha) = \mathrm{id}(\alpha) + 1\} \in U$, so $(f_1) = (\mathrm{id}) + j(1) = (\mathrm{id}) + 1$. Similarly, taking $f_2 : \alpha \mapsto \alpha + \omega_1$ and $f_3 : \alpha \mapsto \alpha \cdot 2$ we may show that $(f_2) = (\mathrm{id}) + \omega_1$ while $(f_3) = (\mathrm{id}) \cdot 2$.

Now fix $\xi < \kappa$ and take $f : \kappa \to \kappa$ with $f(\alpha) := \xi$ if α is even and $f(\alpha) := \alpha$ if α is odd. Let $E := \{\alpha < \kappa : \alpha \text{ is even}\} \subseteq \kappa$, and $O := \kappa \setminus E$. By regularity of κ we can see that both E and O have size κ, so either one could be in U. If $E \in U$ then $(f) = j(\xi) = \xi < (\mathrm{id})$, while if $O \in U$ then $(f) = (\mathrm{id})$. In either case, $(f) \leq (\mathrm{id})$. $\dashv$

(34) Prove that if $U \in M$, then there is a surjection from κ^κ onto $j(\kappa)$ in M. Deduce that $U \notin M$.

Proof. Recall that $j(\kappa) = \{(f)_U : f \in \kappa^\kappa\}$. Hence, the map $\Phi : \kappa^\kappa \to j(\kappa)$ given by $f \mapsto (f)$ is a surjection from κ^κ onto $j(\kappa)$. Now, suppose that $U \in M$. Then the relation $\sim_U \subseteq \kappa^\kappa \times \kappa^\kappa$ given by

$$f \sim_U g \iff \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U$$

is in M by Separation. Hence, for every $f \in \kappa^\kappa$, $[f]_U \in M$ by Separation. In particular, $\{[f]_U : f \in \kappa^\kappa\} \in M$ by Replacement. Hence, the map $\Phi_1 : f \mapsto [f]_U$ is in M by Separation. Since the Mostowski collapsing function is absolute between transitive models of $\mathsf{ZF}^- - \mathsf{P}$, it follows that the map $\Phi_2 : [f]_U \mapsto (f)_U$ is also in M. Consequently, $\Phi = \Phi_2 \circ \Phi_1 \in M$, and so $M \models |j(\kappa)| \leq 2^\kappa$, contradicting that $M \models$ "$j(\kappa)$ is a strong limit". $\square$

(35) Prove that U is normal ultrafilter if and only if $(\mathrm{id}) = \kappa$.

Suppose that U is normal, and let f be arbitrary such that $(f) \in (\mathrm{id})$, i.e. $S := \{\xi < \kappa : f(\xi) < \xi\} \in U$. In particular, S is U-stationary and $f : S \to \kappa$ is regressive. It follows by normality and (29) that there exists some $\alpha < \kappa$ such that $f^{-1}(\{\alpha\}) \in U$. Consequently, $(f) = j(\alpha)$. Since f was arbitrary, we obtain that $(\mathrm{id}) \leq \kappa$, and so $(\mathrm{id}) = \kappa$.

Conversely, assume that $(\mathrm{id}) = \kappa$. Fix an arbitrary $X \in U$ and a regressive function $f : X \to \kappa$. It follows that $(f) < (\mathrm{id}) = \kappa$, and so there is some $\alpha < \kappa$ such that $(f) = \alpha = j(\alpha)$. Consequently, $\{\xi < \kappa : f(\xi) = \alpha\} \in U$, i.e. $f^{-1}(\alpha) \in U$. Since X and f were arbitrary, (29) implies that U is normal as required.

(36) Presentation Example. Assume that $M \models$"κ is measurable" witnessed by an ultrafilter $(g) \in M$ and $(f) = \kappa$. Use this to prove directly (without using a reflection argument) that there are κ many measurable cardinals below κ.

[We discussed "alternative reflection theorems" in the lectures which were similar, but used a normal ultrafilter U which we do not assume here.]

Solution. Let $(g) \in M$ and $(f) = \kappa$ be such that $M \models$"(g) is a non-principal (f)-complete ultrafilter on (f)". It follows by Los that

$$\{\alpha < \kappa : g(\alpha) \text{ is a non-principal } f(\alpha)\text{-complete ultrafilter on } f(\alpha)\} \in U.$$

Fix some $\gamma < \kappa$. Since $\gamma = j(\gamma) < \kappa$ it follows that $\{\alpha < \kappa : \gamma < f(\alpha)\} \in U$. In particular, there is some $\alpha < \kappa$ with $f(\alpha) > \gamma$ and $g(\alpha)$ a $f(\alpha)$-complete non-principal ultrafilter on $f(\alpha)$. Thus, $f(\alpha)$ is a measurable cardinal strictly between γ and κ. Since γ was arbitrary, this implies that the set of measurables below κ is unbounded in κ, and so by regularity, it must have size κ. ⊣

(37) Let U be a κ-complete nonprincipal ultrafilter on κ. Show that if $\{\alpha + 1\, ;\, \alpha < \kappa\} \in U$, then U cannot be normal. Use this to show that if κ is measurable, then there are κ-complete nonprincipal ultrafilters on κ that are not normal.

[*Hint.* Use the fact that there is a bijection between $\{\alpha + 1\, ;\, \alpha < \kappa\}$ and κ.]

Solution. Suppose that $S := \{\alpha + 1 : \alpha < \kappa\} \in U$. Then $A_\alpha := \{x < \kappa : \alpha + 1 < x\} \in U$, since its complement has size $< \kappa$ and U is κ-complete. Consider $\Delta_{\alpha<\kappa} A_\alpha$. It follows that this cannot contain any successor ordinals, as if $\beta + 1 \in \Delta_{\alpha<\kappa} A_\alpha$ then $\beta + 1 \in \bigcap_{\alpha<\beta+1} A_\alpha = A_\beta$, contradiction. Thus, $\Delta_{\alpha<\kappa} A_\alpha \cap S = \emptyset$, and so $\Delta_{\alpha<\kappa} A_\alpha \notin U$, i.e. U is not normal.

So, assume that κ is measurable, and fix a bijection $f : \kappa \to S$ and a κ-complete non-principal ultrafilter U on κ. Let $g : \kappa \to \kappa$ be the composition of f with the inclusion map $S \hookrightarrow \kappa$. It follows by (23) that $g_\star U$ is a κ-complete ultrafilter on κ, which moreover is easily seen to be non-principal. Finally $S \in g_\star U$ and so the fact above implies that $g_\star U$ is not normal. ⊣

(38) Assume that $\kappa < \lambda$ are ordinals and U is an ultrafilter on κ that is not ω_1-complete. Prove that $\mathbf{V}_\lambda^\kappa/U$ is illfounded.

[*Remark.* In the lectures, we proved the converse. So, together, we have that U is ω_1-complete if and only if the ultrapower is wellfounded.]

Solution. If U is not ω_1-complete, then there are sets $\{X_n : n \in \omega\} \subseteq U$ such that $\bigcap_{n\in\omega} X_n \notin U$. For each $k \in \omega$ define, $g_k : S \to V$ by $g_k(i) = n - k$ if $i \in (\bigcap_{m<n} X_m) \setminus X_n$ and $n \geq k$, and otherwise $g_k(i) = 0$. It follows that

$$\{i \in S : g_{k+1}(i) \in g_k(i)\} \supseteq \bigcap_{m\leq k} X_m \setminus \bigcap_{n\in\omega} X_n \in U$$

for every $k \in \omega$, and so $\langle (g_n) : n \in \omega \rangle$ confirms that $\mathbf{V}_\lambda^\kappa/U$ is illfounded. ⊣

(39) Assume that $\kappa < \lambda$ are ordinals, and U is a *principal ultrafilter* on κ. Form the ultrapower and its transitive Mostowski collapse M as in the case of nonprincipal ultrafilters and prove that $M = \mathbf{V}_\lambda$.

Solution. Suppose that U is principal, i.e. there is some $\alpha < \kappa$ such that $\{\alpha\} \in U$. We argue that the ultrapower embedding is surjective. Indeed, let $f : \kappa \to \mathbf{V}_\lambda$ be an arbitrary function, and let $s := f(\alpha)$. Then $\{\beta < \kappa : f(\beta) = s\} \supseteq \{\alpha\} \in U$, implying that $[f]_U = [\mathrm{const}_s]$. Consequently, the ultrapower embedding is an isomorphism. Finally, the transitivity of $\mathbf{V}_\lambda$ implies that it is equal to its Mostowski collapse, and thus $M = \mathbf{V}_\lambda$. ⊣

(40) Presentation Example. Suppose that κ is a measurable cardinal and U is a κ-complete ultrafilter on κ, and $\pi : \mathbf{V}_\kappa \to \mathrm{Ult}(\mathbf{V}_\kappa, U)$ is the ultrapower embedding, i.e., $\pi(x) := [\mathrm{c}_x]_U$. By Łoś's Theorem, π is an elementary embedding. Show that $\{\pi(x)\, ;\, x \in \mathbf{V}_\kappa\}$ is isomorphic to $\mathbf{V}_\kappa$ and transitive in $\mathrm{Ult}(\mathbf{V}_\kappa, U)$, i.e., if $z \in \pi(x)$, then there is $y \in \mathbf{V}_\kappa$ such that $z = \pi(y)$.

Conclude that the order type of the ordinals of $\mathrm{Ult}(\mathbf{V}_\kappa, U)$ is not equal to κ and that therefore $\mathrm{Ult}(\mathbf{V}_\kappa, U)$ is not isomorphic to $\mathbf{V}_\kappa$.

(41) Prove that the Mitchell order on normal ultrafilters is a transitive relation, i.e., if U_0, U_1, and U_2 are normal ultrafilters, $U_0 < U_1$, and $U_1 < U_2$, then $U_0 < U_2$.

Solution. Let U_0, U_1, U_2 be κ-complete normal non-principal ultrafilters on κ, and assume that $U_0 < U_1$ and $U_1 < U_2$. We wish to show that $U_0 < U_2$. By the Mitchell lemma, this is equivalent to finding some $X \in U_2$ and $\{V_\alpha : \alpha \in X\}$ such that V_α is a normal α-complete non-principal ultrafilter on α, and for all $Z \subseteq \kappa$

$$Z \in U_0 \iff \{\alpha \in X : Z \cap \alpha \in V_\alpha\} \in U_2.$$

Our assumptions on U_1 and U_2 guarantee the existence of $X_1 \in U_1$, $\{U^1_\alpha : \alpha \in X_1\}$, and $X_2 \in U_2$, $\{U^2_\alpha : \alpha \in X_2\}$ satisfying the conditions of the Mitchell lemma. Letting $X := \{\beta \in X_2 : X_1 \cap \beta \in U^2_\beta\}$, we see that $X_1 \in U_1$ implies that $X_2' \in U_2$. Moreover, define for each $\beta \in X$ the sets $V_\beta \subseteq \mathcal{P}(\beta)$ by

$$S \in V_\beta \iff \{\alpha \in X_1 \cap \beta : S \cap \alpha \in U^1_\alpha\} \in U^2_\beta.$$

It is easy to see that each V_β is closed under supersets and intersections of size $< \beta$. Moreover,

$$S \notin V_\beta \implies \{\alpha \in X_1 \cap \beta : S \cap \alpha \in U^1_\alpha\} \notin U^2_\beta \implies$$

$$\{\alpha \in X_1 \cap \beta : S \cap \alpha \notin U^1_\alpha\} \in U^2_\beta \implies \{\alpha \in X_1 \cap \beta : (\beta \setminus S) \cap \alpha \in U^1_\alpha\} \in U^2_\beta \implies \beta \setminus S \in V_\beta.$$

Hence, each V_β is a β-complete non-principal ultrafilter over β, which can moreover easily be verified to be normal by a similar argument. It follows that for all $Z \subseteq \kappa$:

$$Z \in U_0 \iff \{\alpha \in X_1 : Z \cap \alpha \in U^1_\alpha\} \in U_1 \iff \{\beta \in X_2 : \{\alpha \in X_1 : Z \cap \alpha \in U^1_\alpha\} \cap \beta \in U^2_\beta\} \in U_2$$

$$\iff \{\beta \in X : Z \cap \beta \in V_\beta\} \in U_2,$$

implying that $U_0 < U_2$ as required. $\dashv$

We define by recursion: κ is 0-*measurable* if it is measurable; κ is $\alpha+1$-*measurable* if it α-measurable and there are unboundedly many α-measurables below κ; for a limit ordinal $\lambda \leq \kappa$, κ is λ-*measurable* if it is ξ-measurable for all $\xi < \lambda$.

(42) Prove that if κ is surviving, it is κ-measurable.

Solution. An easy induction argument reveals that if N is a transitive model of ZFC such that $\mathbf{V}_{\kappa+2} \subseteq N$ then "κ is α-measurable" is absolute between $\mathbf{V}$ and N.

So, fix a surviving cardinal κ. We argue by induction that κ is α-measurable for all $\alpha < \kappa$, and hence it is κ-measurable. Clearly, κ is measurable so it is 0-measurable. The non-zero limit stages follow trivially by the induction step. Assume that κ is α-measurable. Then for all $\beta < \kappa$ we have

$$M \models \exists\gamma(\beta < \gamma < j(\kappa) \wedge \gamma \text{ is } \alpha\text{-measurable}).$$

By elementarity this implies that

$$\mathbf{V}_\lambda \models \exists\gamma(\beta < \gamma < \kappa \wedge \gamma \text{ is } \alpha\text{-measurable}).$$

This implies that $\gamma < \kappa$ is indeed α-measurable. Consequently, there are unboundedly many α-measurables below κ, i.e. κ is $\alpha+1$-measurable, as required. $\dashv$

(43) Prove that κ is strongly compact if and only if for every set S, if F is a κ-complete filter on S, there is a κ-complete ultrafilter U extending F.

[*Remark.* In the lectures, we proved the forward direction for $S = \kappa$. *Hint.* For the backward direction, use Example (31).]

Proof. We show the backward direction. Suppose that for any set I, if F if a κ-complete filter on I then there is a κ-complete ultrafilter U extending F. Let S be a set of non-logical symbols, L_S an $\mathrm{L}_{\kappa\kappa}$-language, and $\Phi \subseteq L_S$ κ-satisfiable. Let I be the collection of all subsets of Φ of size $< \kappa$. For every $i \in I$, fix some $M_i \models i$, and let $i^* = \{j \in I : i \subseteq j\}$. Define

$$F := \{X \subseteq I : \exists \alpha < \kappa \text{ and } f : \alpha \to I \text{ such that } \bigcap_{\beta<\alpha} f(\beta)^* \subseteq X\}.$$

Clearly, F is closed under supersets and $< \kappa$-many intersections. Moreover, for every $\alpha < \kappa$ and $f : \alpha \to I$ the set $\bigcap_{\beta<\alpha} f(\beta)^*$ contains $f[\alpha]$ and so it is non-empty. It follows that F is a κ-complete filter on I. By our assumption on κ, there exists a κ-complete ultrafilter U on I extending F. Let $M := \prod_{i\in I} M_i/U$ be the ultraproduct of the M_i over U. We argue that this is a model of Φ. Indeed, let $\phi \in \Phi$ and observe that

$$M \models \phi \iff \{i \in I : M_i \models \phi\} \in U.$$

by Los. The latter is a superset of $\{\phi\}^* = \{i \in I : \{\phi\} \subseteq i\}$ and therefore is in U by construction. It follows that $M \models \phi$ for all $\phi \in \Phi$ as required. $\square$

(44) Let λ be inaccessible and $S \in \mathbf{V}_\lambda$. Suppose that U is an ω_1-complete ultrafilter over S, M the transitive Mostowski collapse of $\mathbf{V}_\lambda^S/U$, and j the ultrapower embedding. Show that for any cardinal μ, we have that M is closed under μ-sequences if and only if $\{j(\alpha)\,;\, \alpha < \mu\} \in M$.

Solution. Let λ, S, U, M be as above. We first argue that if $j[X] \in M$ for some set X and $Y \subseteq M$ is such that $|Y| \leq |X|$ then $Y \in M$. Indeed, enumerate Y by the elements of X as $\{(f_x) : x \in X\}$. Let $h : S \to \mathcal{P}(X)$ be so that $(h) = j[X]$. Define $g : S \to \mathbf{V}_\lambda$ by letting $g(i)$ be the function with domain $h(i)$ such that $g(i)(x) = f_x(i)$. Then $(g)(j(x)) = (f_x)$ for every $x \in X$ and so $\mathrm{Im}((g)) = Y$ as required.

Now, if $j[\mu] \in M$, then letting $f : \mu \to M$ we observe that $f \subseteq M$ and $|f| \leq |\mu| = |j[\mu]|$. Consequently, $f \in M$, and since f was arbitrary we obtain that ${}^{\mu}M \subseteq M$. Conversely, if ${}^{\mu}M \subseteq M$ then $j|_{\mu} \in M$ and so $j[\mu] \in M$ by Separation. $\dashv$

For Examples (45) to (47), let λ be an inaccessible cardinal. If $j\colon \mathbf{V}_\lambda \to M$ is an elementary embedding with critical point κ and $\mu < \lambda$ is any cardinal, we say that j *covers sets of size* μ if for every $X \in [M]^\mu$, there is a $Y \in M$ such that $X \subseteq Y$ and $M \models |Y| < j(\kappa)$.

(45) Prove that if $\mu < j(\kappa)$ and M is closed under μ-sequences, then j covers sets of size μ.

Proof. Assume that $\mu < j(\kappa)$ and M is closed under μ-sequences. Let $X \in [M]^\mu$, and let $f : \mu \to M$ enumerate X. It follows that $f \in M$, and so by Separation $X = f[\mu] \in M$. Clearly, $X \subseteq X$ while $M \models |X| = \mu < j(\kappa)$. $\square$

(46) Prove that if $j\colon \mathbf{V}_\lambda \to M$ is an ultrapower embedding defined from a κ-complete nonprincipal ultrafilter on κ, then j does not cover sets of size κ^+.

Proof. Consider $X := \{j(\alpha) : \alpha < \kappa^+\} \in [M]^{\kappa^+}$. Suppose that X is covered by some $Y \in M$. Then $M \models$ "$Y \cap j(\kappa^+)$ is a cofinal set in $j(\kappa^+)$ of size $< j(\kappa)$", contradicting that $M \models$ "$j(\kappa^+)$ is regular". So j does not cover sets of size κ^+. $\square$

(47) Prove that if $\kappa < \lambda$ is strongly compact, then for each $\mu < \lambda$ there is an elementary embedding $j\colon \mathbf{V}_\lambda \to M$ that covers sets of size μ.

[*Hint.* Let $S := [\mu]^{<\kappa}$ and generate a κ-complete filter on S from the sets $A_\gamma := \{A \in S\,;\, \gamma \in A\}$ (for $\gamma < \mu$). If $X \in [M]^\mu$, say, $X = \{(f_\gamma)\,;\, \gamma < \mu\}$, let $f(A) := \{f_\gamma(A)\,;\, \gamma \in A\}$ and let $Y := (f)$.]

Solution. Let $S := [\mu]^{<\kappa}$, $A_\gamma := \{A \in S\,;\, \gamma \in A\}$, and let

$$F := \{A \subseteq S : \exists \alpha < \kappa \text{ and } f : \alpha \to \mu \text{ such that } \bigcap_{\beta<\alpha} A_{f(\beta)} \subseteq A\}.$$

This is evidently closed under supersets and $< \kappa$ intersections. Moreover, $f[\alpha] \in \bigcap_{\beta<\alpha} A_{f(\beta)}$, and so F is a κ-complete filter. It follows by strong compactness that F extends to a κ-complete ultrafilter U on S. Let M be the Mostowski collapse of the ultrapower $\prod_{A\in S} \mathbf{V}_\lambda / U$, and $j : \mathbf{V}_\lambda \to M$ the ultrapower embedding. We argue that this covers sets of size μ. Indeed, suppose that $X = \{(f_\gamma) : \gamma < \mu\} \in [M]^\mu$. Define $f : S \to \mathbf{V}_\lambda$ by $A \mapsto \{f_\gamma(A) : \gamma \in A\}$, and let $Y := (f)$. Clearly, $Y \in M$ and for all $\gamma < \mu$, $\{A \subseteq S : f_\gamma(A) \in f(A)\} \supseteq A_\gamma \in U$, implying that $(f_\gamma) \in (f)$ by Los. Consequently, $X \subseteq Y$. Finally, $|f(A)| \leq |A| < \kappa$ for every $A \in S$, and so $M \models |Y| < j(\kappa)$ as required. $\dashv$