Example Sheet #3

Examples Classes.
#1: Friday 9 February 2024, 1:30–3:30pm, **MR15**.
#2: Friday 1 March 2024, 1:30–3:30pm, **MR5**.
#3: Friday 15 March 2024, 1:30–3:30pm, **MR5**.

Presentation. Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

Marking. You can submit all of your work to Ioannis Eleftheriadis (`ie257`) as a *single pdf file* by e-mail or hand it to him on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission. Model solutions will be provided on the moodle page of the course.

For Examples (33) to (36), we assume that $\kappa < \lambda$ are measurable and inaccessible, respectively, that U is a κ-complete nonprincipal ultrafilter on κ, that M is the Mostowski collapse of the ultrapower of $\mathbf{V}_\lambda$ modulo U, and that $j : \mathbf{V}_\lambda \to M$ is the ultrapower embedding. We use the notation from the lectures.

(33) Give concrete functions $f : \kappa \to \kappa$ such that $(f) = (\mathrm{id}) + 1$, $(f) = (\mathrm{id}) + \omega_1$, $(f) = (\mathrm{id}) \cdot 2$, $(f) = (\mathrm{id})^+$, and $(f) = (\mathrm{id})^{++}$. Fix $\xi < \kappa$ and consider the function $f(\alpha) := \xi$ if α is even and $f(\alpha) := \alpha$ if α is odd. What can we say about the relation between (id) and (f)?

[*Remark.* As usual, an ordinal α is even if it is of the form $\lambda + 2n$ where λ is a limit ordinal and n is a natural number.]

(34) Prove that if $U \in M$, then there is a surjection from κ^κ onto $j(\kappa)$ in M. Deduce that $U \notin M$.

(35) Prove that U is normal ultrafilter if and only if $(\mathrm{id}) = \kappa$.

(36) Presentation Example. Assume that $M \models$"κ is measurable" witnessed by an ultrafilter $(g) \in M$ and $(f) = \kappa$. Use this to prove directly (without using a reflection argument) that there are κ many measurable cardinals below κ.

[We discussed "alternative reflection theorems" in the lectures which were similar, but used a normal ultrafilter U which we do not assume here.]

(37) Let U be a κ-complete nonprincipal ultrafilter on κ. Show that if $\{\alpha + 1\,;\, \alpha < \kappa\} \in U$, then U cannot be normal. Use this to show that if κ is measurable, then there are κ-complete nonprincipal ultrafilters on κ that are not normal.

[*Hint.* Use the fact that there is a bijection between $\{\alpha + 1\,;\, \alpha < \kappa\}$ and κ.]

(38) Assume that $\kappa < \lambda$ are ordinals and U is an ultrafilter on κ that is not ω_1-complete. Prove that $\mathbf{V}_\lambda^\kappa/U$ is illfounded.

[*Remark.* In the lectures, we proved the converse. So, together, we have that U is ω_1-complete if and only if the ultrapower is wellfounded.]

(39) Assume that $\kappa < \lambda$ are ordinals, and U is a *principal ultrafilter* on κ. Form the ultrapower and its transitive Mostowski collapse M as in the case of nonprincipal ultrafilters and prove that $M = \mathbf{V}_\lambda$.

(40) Presentation Example. Suppose that κ is a measurable cardinal and U is a κ-complete ultrafilter on κ, and $\pi : \mathbf{V}_\kappa \to \mathrm{Ult}(\mathbf{V}_\kappa, U)$ is the ultrapower embedding, i.e., $\pi(x) := [\mathrm{c}_x]_U$. By Łoś's Theorem, π is an elementary embedding. Show that $\{\pi(x)\,;\, x \in \mathbf{V}_\kappa\}$ is isomorphic to $\mathbf{V}_\kappa$ and transitive in $\mathrm{Ult}(\mathbf{V}_\kappa, U)$, i.e., if $z \in \pi(x)$, then there is $y \in \mathbf{V}_\kappa$ such that $z = \pi(y)$.

Conclude that the order type of the ordinals of $\mathrm{Ult}(\mathbf{V}_\kappa, U)$ is not equal to κ and that therefore $\mathrm{Ult}(\mathbf{V}_\kappa, U)$ is not isomorphic to $\mathbf{V}_\kappa$.

(41) Prove that the Mitchell order on normal ultrafilters is a transitive relation, i.e., if U_0, U_1, and U_2 are normal ultrafilters, $U_0 < U_1$, and $U_1 < U_2$, then $U_0 < U_2$.

We define by recursion: κ is 0-*measurable* if it is measurable; κ is $\alpha+1$-*measurable* if it α-measurable and there are unboundedly many α-measurables below κ; for a limit ordinal $\lambda \leq \kappa$, κ is λ-*measurable* if it is ξ-measurable for all $\xi < \lambda$.

(42) Prove that if κ is surviving, it is κ-measurable.

(43) Prove that κ is strongly compact if and only if for every set S, if F is a κ-complete filter on S, there is a κ-complete ultrafilter U extending F.

[*Remark.* In the lectures, we proved the forward direction for $S = \kappa$. *Hint.* For the backward direction, use Example (31).]

(44) Let λ be inaccessible and $S \in \mathbf{V}_\lambda$. Suppose that U is an ω_1-complete ultrafilter over S, M the transitive Mostowski collapse of $\mathbf{V}_\lambda^S/U$, and j the ultrapower embedding. Show that for any cardinal μ, we have that M is closed under μ-sequences if and only if $\{j(\alpha)\,;\, \alpha < \mu\} \in M$.

For Examples (46) to (48), let λ be an inaccessible cardinal. If $j\colon \mathbf{V}_\lambda \to M$ is an elementary embedding with critical point κ and $\mu < \lambda$ is any cardinal, we say that j *covers sets of size* μ if for every $X \in [M]^\mu$, there is a $Y \in M$ such that $X \subseteq Y$ and $M \models |Y| < j(\kappa)$.

(46) Prove that if $\mu < j(\kappa)$ and M is closed under μ-sequences, then j covers sets of size μ.

(47) Prove that if $j\colon \mathbf{V}_\lambda \to M$ is an ultrapower embedding defined from a κ-complete nonprincipal ultrafilter on κ, then j does not cover sets of size κ^+.

(48) Prove that if $\kappa < \lambda$ is strongly compact, then for each $\mu < \lambda$ there is an elementary embedding $j\colon \mathbf{V}_\lambda \to M$ that covers sets of size μ.

[*Hint.* Let $S := [\mu]^{<\kappa}$ and generate a κ-complete filter on S from the sets $A_\gamma := \{A \in S\,;\, \gamma \in A\}$ (for $\gamma < \mu$). If $X \in [M]^\mu$, say, $X = \{(f_\gamma)\,;\, \gamma < \mu\}$, let $f(A) := \{f_\gamma(A)\,;\, \gamma \in A\}$ and let $Y := (f)$.]