Model Solutions for Example Sheet #2

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Useful lemma: For every measure $\mu : \mathcal{P}(S) \to [0,1]$, if $T \subseteq \{X \subseteq S : \mu(X) > 0\}$ is an uncountable family of subsets of S of positive measure, then there are two distinct sets $X, Y \in T$ such that $\mu(X \cap Y) > 0$.

Proof. If T is uncountable then since $T = \cup_{n=1}^{\infty}\{X \in T : \mu(X) > \frac{1}{n}\}$ it follows that there is some $k \geq 1$ such that $\{X \in T : \mu(X) > \frac{1}{k}\}$ is uncountable. Consider an enumeration $X_0, X_1, X_2, \ldots$ of these sets, and define by recursion:

$$A_0 = X_0; A_{n+1} = X_{n+1} \setminus (\bigcup_{i \in n+1} A_i).$$

This is a family of pairwise disjoint subsets of X. Now, assuming that for all $X, Y \in T$ the measure of $X \cap Y$ is 0, we easily obtain by induction that

$$\mu(A_n) = \mu(X_n)$$

by the additivity of μ. It follows that

$$\mu(\bigcup_{n\in\omega} A_n) = \sum_{n\in\omega} \mu(A_n) = \sum_{n\in\omega} \mu(X_n) > 1,$$

as μ is σ-additive and each X_n has measure $> \frac{1}{k}$. This contradicts that μ maps into $[0,1]$, and therefore there must exist $X, Y \in T$ with $\mu(X \cap Y) > 0$. □

(17) Assume that there is an set with a Banach measure on it and let κ be the smallest cardinality of such a set. Prove that every Banach measure on κ is κ-additive.

Solution. Let μ be a Banach measure over κ, and assume for a contradiction that it is not κ-additive. It follows that there is some $\gamma < \kappa$, and $\{X_\alpha : \alpha < \gamma\} \subseteq \mathcal{P}(\kappa)$ pairwise disjoint sets such that

$$\mu(\bigcup_{\alpha<\gamma} X_\alpha) \neq \sum_{\alpha<\gamma} \mu(X_\alpha).$$

By σ-additivity it follows that $\gamma > \omega$, while the useful lemma above implies that there are at most countable many $\alpha < \gamma$ with $\mu(X_\alpha) > 0$. Removing these using σ-additivity, we may assume without loss of generality that $\mu(X_\alpha) = 0 \forall \alpha < \gamma$, but $\mu(\cup_{\alpha<\gamma} X_\alpha) = r > 0$. We then define $\bar{\mu} : \mathcal{P}(\gamma) \to [0,1]$ by

$$\bar{\mu}(Y) = \frac{\mu(\bigcup_{\alpha \in Y} X_\alpha)}{r}.$$

An easy check reveals that this is a Banach measure over γ, contradicting the minimality of κ. ⊣

(18) If μ is a Banach measure on S, we say that $A \subseteq S$ is an *atom of* μ if $\mu(A) > 0$ and for each $B \subseteq A$, either $\mu(B) = \mu(A)$ or $\mu(B) = 0$. We call μ *atomless* if it does not have any atoms. Prove that if μ is atomless, then for each set $A \subseteq S$, there is some $B \subseteq A$ such that $\mu(B) = \frac{1}{2} \cdot \mu(A)$.

Solution. Let μ be an atomless Banach measure on S. We first argue that for every > 0 and every $X \subseteq S$ with $\mu(X) > 0$ there exists some $Y \subseteq X$ such that $0 < \mu(Y) < \varepsilon$. Towards this, it suffices to construct an $\subseteq$-descending sequence with $0 < \mu(X_i) \leq \frac{1}{2} \cdot \mu(X_i)$ for all $i \in \omega$. Indeed, given X_i, since

μ is atomless there exist disjoint sets $A, B \subseteq X_i$ such that $A \cup B = X_i$ and $0 < \mu(A) \leq \mu(B) < \mu(X_i)$. We thus let $X_{i+1} := A$ and continue in this way.

Now, given $A \subseteq X$ define a $\subseteq$-descending sequence with $Y_0 = A$, and given X_α, if $\mu(X_\alpha) > \frac{1}{2}\mu(A)$ then we use the claim above to find some $X_{\alpha+1} \subseteq X_\alpha$ such that

$$\mu(X_\alpha) > \mu(X_{\alpha+1}) \geq \frac{1}{2}\mu(A).$$

Finally for limit ordinals λ we let $Y_\lambda = \cap_{\alpha<\lambda} X_\alpha$. If at any stage the sequence can no longer extended then we have produce some X_α such that $\mu(X_\alpha) = \frac{1}{2}\mu(A)$. Otherwise, the family $\{X_\alpha \setminus X_{\alpha+1} : \alpha < \omega_1\}$ is uncountable. Hence, if $\mu(X_\alpha \setminus X_{\alpha+1}) > 0$ for all α then by the useful lemma above there exist $\beta \neq \gamma$ such that

$$\mu(X_\beta \setminus X_{\beta+1} \cap X_\gamma \setminus X_{\gamma+1}) > 0,$$

contradiction. ⊣

(19) Assume that there is a κ-additive atomless Banach measure on κ. Prove that $\kappa \leq 2^{\aleph_0}$. Derive that if there is a real-valued measurable cardinal κ with an atomless κ-additive measure on it, then there are weakly inaccessible cardinals that are not inaccessible and CH is false.

Solution. For each finite sequence $s \in {}^{<\omega}\omega$ we define $X_s \subseteq \kappa$ recursively: we first let $X_\emptyset = \kappa$, and given X_s, we apply (18) recursively to construct sets $X_{s^\frown\langle i\rangle}$ for each $i \in \omega$ such that $X_s = \cup_{i\in\omega} X_{s^\frown\langle i\rangle}$ is a disjoint union and $\mu(X_{s^\frown\langle i\rangle}) = 2^{-(i+1)} \cdot \mu(X_s)$.

For each $f \in {}^\omega\omega$ set $Y_f = \cap_{n\in\omega} X_{f|_n}$; in particular $\mu(Y_f) = 0$. It follows that

$$S = \bigcup\{Y_f : f \in {}^\omega\omega\},$$

and so μ cannot be $(2^{\aleph_0})^+$-additive. Hence $\aleph_0 < \kappa \leq 2^{\aleph_0}$ as required.

Now, since real-valued measurable cardinals are weakly inaccessible, the above implies that κ is weakly inaccessible but not strong limit; hence it is not strongly inaccessible. We now distinguish two cases. If $\kappa = 2^{\aleph_0}$ then since it is a limit, we obtain some $\aleph_0 < \lambda < 2^{\aleph_0}$ and hence CH fails. On the other hand if $\kappa < 2^{\aleph_0}$ then $\aleph_0 < \kappa < 2^{\aleph_0}$ and CH fails once again. ⊣

(20) Show that if μ is a Banach measure on S that has an atom, then there is a two-valued Banach measure on S.

Solution. Consider an atom $A \subseteq S$, and let $m : \mathcal{P}(S) \to \{0, 1\}$ be given by

$$m(X) = \frac{\mu(X \cap A)}{\mu(A)}.$$

An easy check reveals that this is non-trivial, σ-additive, and two-valued as required. ⊣

(21) Let U be an ultrafilter on κ. Show that U is λ-complete if and only if for each $\gamma < \lambda$ and $\{A_\alpha\,;\, \alpha < \gamma\} \subseteq U$, we have that $\bigcap_{\alpha<\gamma} A_\alpha \neq \varnothing$.

Solution. Clearly, if U is λ-complete then for all $\gamma < \lambda$ and $\{A_\alpha : \alpha < \gamma\} \subseteq U$ it holds that $\bigcap A_\alpha \in U$, and hence $\bigcap A_\alpha \neq \emptyset$. Conversely, suppose that U is not λ-complete. Then there is some $\gamma < \lambda$ and a family $\{A_\alpha : \alpha < \gamma\} \subseteq U$ such that $A := \bigcap A_\alpha \notin U$. Consequently, $\kappa \setminus A \in U$. It follows that the family $\{A_\alpha : \alpha < \gamma\} \cup \{\kappa \setminus A\}$ contains elements of U, and $\bigcap A_\alpha \cap \kappa \setminus A = \emptyset$ as required. ⊣

(22) Using the Axiom of Choice, show that every filter can be extended to an ultrafilter

Solution. Let F be a non-principal filter on a set X. Consider $S = \{U \subseteq \mathcal{P}(X) : U$ is a non-principal filter and $F \subseteq U\}$, which is a set by separation. Let $P = (X_i)_{i\in I}$ be a chain (w.r.t inclusion) in S, and consider $\bigcup P$. This is still a filter on X. Indeed $X \in \bigcup P$, while if $\emptyset \in \bigcup P$ then $\emptyset \in X_i$

for some i, contradiction. It is clearly upwards closed, while if $A, B \in \bigcup P$ then we may find some i such that $A, B \in X_i$ and so $A \cap B \in X_i \subseteq \bigcup P$. Since $\{a\} \in \bigcup P$ implies $\{a\} \in X_i$ for some i, this is a non-principal filter extending F so $\bigcup P \in S$. By Zorn's lemma, S therefore has a maximal element U. This is an ultrafilter: Suppose that $A \subseteq X$ is such that $A \notin U$ and $X \setminus A \notin U$, and take some $Y \in U$. If $Y \cap A = \emptyset$, then $Y \subseteq X \setminus A$ so $X \setminus A \in U$, contradiction. So the set $\{Z \subseteq X : Z \supseteq Y \cap A \text{ or } Z \supseteq Y, Y \in U\}$ is an ultrafilter extending U, contradicting its maximality. Note that $U \in S$ so by definition it is non-principal. $\dashv$

(23) If C is a set of subsets of Z, we say that D is the *collection generated by* C if D is minimal such that $C \subseteq D \subseteq \wp(Z)$ and D is closed under finite intersections and supersets.

Let X and Y be sets, $f : X \to Y$ a function, F a filter on X and G a filter on Y. Let f_*F be the collection generated by $\{f[A] \,;\, A \in F\}$ (called the *pushout of* F) and f^*G be the collection generated by $\{f^{-1}[B] \,;\, B \in G\}$ (called the *pullback of* G).

(a) Under which conditions on f are f_*F or f^*G filters?

(b) Under which conditions on f are $\{f[A] \,;\, A \in F\}$ or $\{f^{-1}[B] \,;\, B \in G\}$ filters?

(c) If F or G are ultrafilters, are f_*F or f^*G?

(d) If F or G are κ-complete, are f_*F or f^*G?

(e) If F or G are nonprincipal, are f_*F or f^*G?

Solution. Firstly, note that since $f[A \cap B] \subseteq f[A] \cap f[B]$, it follows that $f_*F = \{S \subseteq Y; f[A] \subseteq S, A \in F\} = \{S \subseteq Y; f^{-1}[S] \in F\}$. Moreover, since $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$, $f^*G = \{S \subseteq X : f^{-1}[B] \subseteq S, B \in G\}$. Provided that f is surjective, this implies that $f^*G \subseteq \{S \subseteq X : f[S] \in G\}$. So:

(a) f_*F is always a filter. Clearly $\emptyset \notin f_*F$, while it is trivially closed under finite intersections and supersets. f^*G is a filter provided that f is surjective. Indeed, in that case f^*G does not contain the empty set, and hence it is a filter.

(b) $\{f[A] \,;\, A \in F\}$ is a filter provided that f is surjective. Indeed, in that case it is closed under supersets, while $f[A \cap B] \subseteq f[A] \cap f[B]$ implies that it is closed under finite unions. Clearly $\emptyset \neq f[A]$ for any $A \in U$, so this is a filter. On the other hand, surjectivity does not suffice to ensure that $\{f^{-1}[B] \,;\, B \in G\}$ is a filter. For instance, take $f : \omega \to \omega$ given by $0 \mapsto 0, n+1 \mapsto n$, and let G be a principal ultrafilter on ω focusing on 1. Then $\{f^{-1}[B] \,;\, B \in G\}$ is not closed under supersets, as $\{2\} = f^{-1}[\{1\}]$ but $\{1, 2\} \neq f^{-1}[S]$ for any $S \in G$. However, if f is bijective then f^*G is clearly a filter.

(c) Assuming that F is an ultrafilter, then so is f_*F. Indeed, $S \notin f_*F$ implies that $f^{-1}[S] \notin F$, and so $X \setminus f^{-1}[S] \in F$. But then $f[X \setminus f^{-1}[S]] \subseteq Y \setminus S \in f_*F$, and so $Y \setminus S \in f_*F$.

(d) If F is κ-complete, then so is f_*F since $f^{-1}[\bigcap A_i] = \bigcap f^{-1}[A_i]$. Likewise, if G is κ-complete and f is surjective then f^*G is κ-complete.

(e) Even if F is non-principal, f_*F can still be principal, e.g. if f is constant. On the other hand f^*G is non-principal if G is also non-principal and f is surjective. Indeed, in this case $|f^{-1}[X]| \geq |X|$ and so if f^*G contains a singleton then so must G.

$\dashv$

(24) Presentation Example. A cardinal κ is called an *Ulam cardinal* if there is an $\aleph_1$-complete non-principal ultrafilter on κ. Show that the smallest Ulam cardinal is a measurable cardinal.

Solution. Let κ be the least Ulam cardinal, and let U be an $\aleph_1$-complete non-principal ultrafilter on κ. Suppose that U is not κ-complete. We may therefore find a partition $\{X_\alpha : \alpha < \gamma\}$ of κ with $\gamma < \kappa$, such that $X_\alpha \notin U$ for all $\alpha < \gamma$. Define a surjection $f : \kappa \to \gamma$ by $f(x) = \alpha$ if and only if $x \in X_\alpha$. This induces the pushout filter f_*U on γ given by $Z \in F$ if and only if $f^{-1}(Z) \in U$. This is an $\aleph_1$-complete ultrafilter by Question 20, while it is also non-principal. Indeed, if $\{\alpha\} \in F$ for some $\alpha < \gamma$ then $f^{-1}(\alpha) = X_\alpha \in U$, contradiction. Hence $\gamma < \kappa$ is Ulam, contradicting minimality. It follows that U is κ-complete, and so κ is measurable. ⊣

(25) Show that the Erdős arrow notation is stable under increasing numbers on the left hand side of the arrow and decreasing numbers on the right hand side of the arrow. I.e., if $\kappa \to (\lambda)^m_\mu$ and $\kappa' \geq \kappa$, $\lambda' \leq \lambda$, $\mu' \leq \mu$, and $m' \leq m$, then $\kappa' \to (\lambda')^{m'}_{\mu'}$.

Solution. Suppose that $\kappa \to (\lambda)^m_\mu$ and let $\kappa' \geq \kappa$, $\lambda' \leq \lambda$, $\mu' \leq \mu$, and $m' \leq m$. Let $\chi : [\kappa']^{m'} \to \mu'$ be a μ'-colouring of the subsets of κ' of size m'. Define $\xi : [\kappa]^m \to \mu$ by mapping a subset $\{a_1, \ldots, a_m\}$ of κ with $a_i < a_j$ for $i < j$ to $\chi(\{a_1, \ldots, a_{m'}\})$. It follows that there is a set $X \subseteq \kappa$ of size λ such that ξ is constant on $[X]^m$. Letting X' be any subset of X of size λ', this implies that χ is constant on $[X']^{m'}$. Since χ was arbitrary, this shows that $\kappa' \to (\lambda')^{m'}_{\mu'}$. ⊣

(26) Let κ be regular and $\lambda < \kappa$. Let $2^\lambda := \{f\,;\, f : \lambda \to \{0,1\}\}$ be ordered lexicographically by $f <_{\text{lex}} g$ if $f(\alpha) = 0$ and $g(\alpha) = 1$ if α is the least ordinal where f and g differ. Show that $(2^\lambda, <_{\text{lex}})$ has no strictly increasing or decreasing sequences of length κ.

Solution. Assume for a contradiction that there is a set $\{f_\alpha : \alpha < \kappa\} \subseteq 2^\lambda$ of size κ such that $f_i <_{\text{lex}} f_j$ for $i < j$. Let $\gamma \leq \lambda$ be least such that $\{f_\alpha \restriction_\gamma : \alpha < \kappa\}$ has size κ, and reorder appropriately so that $f_i \restriction_\gamma <_{\text{lex}} f_j \restriction_\gamma$ for all $i < j$ from κ. Define $g : \kappa \to \gamma$ by mapping α to the least ordinal $\xi_\alpha < \gamma$ such that $f_\alpha \restriction_{\xi_\alpha} = f_{\alpha+1} \restriction_{\xi_\alpha}$, and $f_\alpha(\xi_\alpha) = 0, f_{\alpha+1}(\xi_\alpha) = 1$. By regularity of κ, it follows that there is a set $X \subseteq \kappa$ of size κ such that f is constant on X; κ is the union of the pre-images of singletons, and as there are γ many such pre-images one must have size κ by regularity. Let $c < \gamma$ be such that $g(x) = c$ for all $x \in X$. We argue that the set $\{f_\alpha \restriction_c : \alpha < \kappa\}$ contains pairwise distinct elements. Indeed, if $\alpha < \beta$ are in X then $f_\alpha \restriction_c = f_\beta \restriction_c$ implies that $f_\alpha(x) = f_{\alpha+1}(x) = f_\beta(x)$ for $x < c$, and $f_\alpha(c) = f_\beta(c) = 0, f_{\alpha+1}(c) = 1$. Hence $f_{\alpha+1} >_{\text{lex}} f_\beta$; contradiction.

A similar argument implies that there are no decreasing sequences of length κ in 2^λ. ⊣

(27) Let U be a ultrafilter on κ such that all elements of U have cardinality κ. Show that if U is normal, then U is κ-complete.

Solution. Let $\lambda < \kappa$ and consider a family $\{A_i : i < \lambda\} \subseteq U$. Let $A := \bigcap_{i<\lambda} A_i$. Define the map $f : \kappa \to \kappa$ by

$$f(\alpha) = \begin{cases} \gamma, & \text{if } \alpha \notin A \text{ and } \gamma < \lambda \text{ is the least such that } \alpha \notin A_\gamma; \\ \lambda, & \text{if } \alpha \in A. \end{cases}$$

Since $f(\alpha) \leq \lambda$, it follows that $\{\alpha < \kappa : f(\alpha) \geq \alpha\}$ has size at most λ, and therefore is not in U by our assumption. Hence, $X = \{\alpha < \kappa : f(\alpha) < \alpha\} \in U$. It follows by normality and Question 29 that there is some $\gamma < \kappa$ such that $\{\alpha : f(\alpha) = \gamma\} \in U$. If $\gamma < \lambda$, then $\{\alpha < \kappa : f(\alpha) = \gamma\} \cap A_\gamma = \emptyset \in U$, contradiction. Hence $\gamma = \lambda$, and so $A = \{\alpha < \kappa : f(\alpha) = \lambda\} \in U$. ⊣

(28) Let κ be regular and uncountable. A set $A \subseteq \kappa$ is *closed* if for each limit ordinal $\lambda < \kappa$, if $A \cap \lambda$ is unbounded in λ, then $\lambda \in A$. A set C is called a *club set* (for "closed unbounded") if it is closed and unbounded. Define

$$\mathcal{C} := \{A \subseteq \kappa\,;\, \text{there is a club set } C \subseteq A\}.$$

Show that $\mathcal{C}$ is a κ-complete and normal filter on κ.

Solution. We first argue that for all $\mu < \kappa$ and $\{C_i : i < \mu\}$ such that C_i are club sets in κ, $C := \bigcap_{i<\mu} C_i$ is a club set in κ. Closure follows trivially; if $\lambda < \kappa$ is a limit ordinal such that $\sup(C \cap \lambda) = \lambda$, then $\sup(C_i \cap \lambda) = \lambda$ for all $i < \mu$, and so $\lambda \in C_i$. Hence $\lambda \in C$. We argue that C is unbounded in κ. Fix some $\beta_0 < \kappa$. We define a sequence $\beta_0 < \beta_1 < \ldots$ by induction. Having defined β_n, let $\beta^i_{n+1} \in C_i$ be an ordinal $> \beta_n$, which exists by the unboundedness of C_i. We let $\beta_{n+1} = \sup_{i \in \mu} \beta^i_{n+1}$.

By regularity of κ, it follows that $\beta_{n+1} < \kappa$. Finally, consider $\beta = \sup_{n\in\omega} \beta_n$. Since κ is regular and uncountable, it follows that $\beta < \kappa$. Moreover, $\beta_n < \beta^i_{n+1} < \beta_{n+1}$ for all $n \in \omega$ and $i < \mu$, and so $\beta = \sup_{n\in\omega} \beta_n = \sup_{n\in\omega} \beta^i_n \in C_i$ by closedness of C_i. Hence, $\beta \in C$. Since $\beta_0 < \beta$ and β_0 was arbitrary, this implies that C is unbounded in κ.

We additionally argue that if $\{C_i : i < \kappa\}$ are club sets in κ, then so is $C := \triangle_{i<\kappa} C_i$. Without loss of generality we may assume that $C_i \supseteq C_j$ for $i < j$; indeed, since the intersection of fewer than κ club sets is a club set we may consider the sets $C'_i = \bigcap_{j\leq i} C_i$, which satisfy $\triangle_{i<\kappa} C'_i = \triangle_{i<\kappa} C_i$. So, we first argue that C is closed. Let $\lambda < \kappa$ and suppose that $\sup(C \cap \lambda) = \lambda$. Fix $i < \lambda$, and let $\beta < \lambda$ satisfy $i < \beta$. It follows that there is $\gamma > \beta$ such that $\gamma \in C \cap \lambda$. Since $i < \gamma$ and $\gamma \in C$, this implies that $\gamma \in C_i$. As β was arbitrary, this implies that $\sup(C_i \cap \lambda) = \lambda$. By closedness of C_i, we obtain that $\lambda \in C_i$ for all $i < \lambda$. Hence, $\lambda \in C$ by definition, implying that C is closed. Moreover, if $\alpha < \kappa$ then using the fact that each C_i is unbounded we may recursively construct a sequence $(\beta_n)_{n\in\omega}$ by picking some $\beta_0 > \alpha$ from C_0, and $\beta_{n+1} > \beta_n$ from C_{β_n}. Letting $\beta = \sup_{n\in\omega} \beta_n$, observe that $\beta \in C_{\beta_n}$ for all $n \in \omega$. Indeed, the sets $(C_i)_{i<\kappa}$ form a decreasing chain and each C_i is closed. So, if $i < \beta$ then $i < \beta_n$ for some $n \in \omega$, and so $\beta \in C_{\beta_n} \subseteq C_i$. It follows that $\beta \in C_i$ for all $i < \beta$, implying that $\beta \in C$. Since $\alpha < \beta$ was arbitrary, this implies that C is unbounded.

Finally, it is clear that $\mathcal{C}$ is closed under supersets, while no club set is empty, and so $\emptyset \notin \mathcal{C}$. Together with the above two arguments, this implies that $\mathcal{C}$ is a κ-complete normal filter on κ. ⊣

(29) Let F be a filter on a cardinal κ. Say that for $X \subseteq \kappa$, a function $f : X \to \kappa$ is called *regressive* if $f(\alpha) < \alpha$ for all $0 \neq \alpha \in X$. A set S is called F*-stationary* if for all $X \in F$, we have that $X \cap S \neq \varnothing$. Prove that the following statements are equivalent for a filter F.

(i) The filter F is closed under diagonal intersections.

(ii) For any F-stationary set S and any regressive $f : S \to \kappa$, there is an $\alpha < \kappa$ such that $f^{-1}(\{\alpha\})$ is F-stationary.

Solution. Firstly, observe that a set $S \subseteq \kappa$ is F-stationary if and only if $\kappa \setminus S \notin F$. Clearly, if $\kappa \setminus S \in F$ then S cannot be stationary, while if S is not stationary then $S \cap X = \emptyset$ for some $X \in F$. Consequently, $X \subseteq \kappa \setminus S$, and so $\kappa \setminus S \in F$.

We first argue that $(i) \implies (ii)$. Let S be an F-stationary set and $f : S \to \kappa$ a regressive map such that for all $\alpha < \kappa$ $f^{-1}(\{\alpha\})$ is not F-stationary. Then $A_\alpha := \kappa \setminus f^{-1}(\{\alpha\}) \in F$, and therefore $\triangle_{\alpha<\kappa} A_\alpha = \{\beta \in S : f(\beta) \geq \beta\} = \emptyset$. It follows that F is not closed under diagonal intersections.

Conversely, let $\{A_\alpha : \alpha < \kappa\} \subseteq F$ be such that $\triangle_{\alpha<\kappa} A_\alpha \notin F$. Hence $A := \kappa \setminus \triangle_{\alpha<\kappa} A_\alpha$ is F-stationary. Define $f : A \to \kappa$ by mapping β to the least α such that $\beta \notin A_\alpha$. Clearly, f is regressive. However, $f^{-1}(\{\alpha\}) \cap A_\alpha = \emptyset$ for all $\alpha < \kappa$, and so $f^{-1}(\{\alpha\})$ is not F-stationary for any $\alpha < \kappa$. ⊣

(30) Assume that κ is measurable with a κ-complete nonprincipal ultrafilter U on κ. Use the notation of (29) and let $W := \{f\colon \kappa \to \kappa\,;\, X^f_\alpha \notin U \text{ for all } \alpha < \kappa\}$. Show that there is an $h \in W$ such that for all $f \in W$ we have that $\{\alpha\,;\, h(\alpha) \leq f(\alpha)\} \in U$. Using the notation of (23), show that $h_* U$ is a normal κ-complete nonprincipal ultrafilter on κ.

Proof. Suppose for a contradiction that for all $h \in W$ there is some $h' \in W$ such that

$$\{\alpha < \kappa : h(\alpha) \leq h'(\alpha)\} \notin U.$$

Since U is an ultrafilter this implies that

$$\{\alpha : h'(\alpha) < h(\alpha)\} \in U.$$

So, picking some arbitrary $h_0 \in W$ and letting $h_{n+1} = h'_n$ for all $n \in \omega$, the κ-additivity of U implies that

$$\bigcap_{n\in\omega} \{\alpha < \kappa : h_{n+1}(\alpha) < h_n(\alpha)\} \in U$$

and so in particular it is non-empty. Hence, if α is in this intersection we obtain that $h_0(\alpha) > h_1(\alpha) > h_2(\alpha) > \dots$ is an infinitely decreasing chain in κ, contradicting its well-orderness.

We know by (23) that h_*U is a κ-complete ultrafilter on κ. Moreover, it is non-principal, as if $\{\alpha\} \in h_*U$ for some $\alpha < \kappa$ then $h^{-1}(\alpha) = X^h_\alpha \in U$, contradicting that $h \in W$. We argue that h_*U is normal. If not, then by (29) there is some h_*U-stationary set $S \subseteq \kappa$ and a regressive function $f : S \to \kappa$ such that for all $\alpha < \kappa$ the set X^f_α is not h_*U-stationary. Since h_*U is an ultrafilter this implies that $X^f_\alpha \notin h_*U$, and hence $X^{f\circ h}_\alpha \notin U$. Since this holds for all $\alpha < \kappa$, we obtain that $f \circ h \in W$. The choice of h thus implies that

$$\{\alpha < \kappa : h(\alpha) \leq f \circ h(\alpha)\} \in U, \text{and so}$$
$$\{\gamma < \kappa : \gamma \leq f(\gamma)\} \in h_*U\}.$$

However, since f is regressive on S we obtain that

$$S \cap \{\gamma < \kappa : \gamma \leq f(\gamma)\} = \emptyset,$$

contradicting that S is h_*U-stationary. □

(31) Presentation Example. Assume that κ is measurable with a κ-complete nonprincipal ultrafilter U on κ. Formulate and prove Łoś's Theorem for $\mathcal{L}_{\kappa\kappa}$-languages for the ultrapowers by U.

Solution. Łoś's Theorem for $\mathcal{L}_{\kappa\kappa}$-languages: Suppose that U is a κ-complete ultrafilter over κ, M_α an L_S-structure for all $\alpha < \kappa$, and $M := \prod_{\alpha<\kappa} M_\alpha/U$ the ultraproduct of the M_α over U. Then for every formula $\phi \in L_S$ we have

$$M \models \phi([\bar{f}]) \iff \{\alpha : M_\alpha \models \phi(\bar{f}(\alpha))\} \in U.$$

We show this by induction on the structure of $\mathcal{L}_{\kappa\kappa}$-formulas. For atomic formulas this follows by the standard version of Łoś's Theorem, while propositional connectives work as in the classical case too. So, suppose that $\phi := \bigwedge_{\xi<\lambda} \phi_\xi$. It follows that

$$M \models \phi([\bar{f}]) \iff \forall\xi < \lambda\ M \models \phi_\xi([\bar{f}]) \iff \forall\xi < \lambda\ X_\xi := \{\alpha < \kappa : M_\alpha \models \phi_\xi(\bar{f}(\alpha))\} \in U,$$

where the last bi-implication follows by the induction hypothesis. Moreover, the κ-completeness and upwards closure of U implies that the last condition is equivalent to

$$\{\alpha < \kappa : \forall\xi < \lambda\ M_\alpha \models \phi_\xi(\bar{f}(\alpha))\} \in U \iff \{\alpha < \kappa : M_\alpha \models \bigwedge_{\xi<\lambda} \phi_\xi(\bar{f}(\alpha))\} \in U,$$

as required. The case $\phi(\bar{x}) := \exists^\lambda \bar{y}\psi(\bar{x}, \bar{y})$ follows similarly. ⊣

(32) Let S be a set of symbols for an $\mathcal{L}_{\kappa\kappa}$ language L_S. Show that if $|S| \leq \kappa$, then $|L_S| = \kappa$. Use this and (31) to give a alternative proof of the fact that every measurable cardinal is weakly compact.

[*Hint.* Use the characterisation—not proved in the course—of weak compactness via compactness of $\mathcal{L}_{\kappa\kappa}$-languages.]

Proof. For the first part, we argue by induction on the structure of $\mathcal{L}_{\kappa\kappa}$-formulas. At the base case, we start with $\leq \kappa$ many atoms, while closing under $\bigwedge_{\alpha<\lambda} \phi_\alpha$ gives

$$\kappa^{<\kappa} = \bigcup_{\lambda<\kappa} \kappa^\lambda = \kappa$$

many formulas, by the fact that κ is regular and a strong limit. For the same reasons, closure under $\exists^\lambda \bar{x}\phi$ gives κ many formulas, and so $|L_S| = \kappa$.

So, let Φ be a set of $\mathcal{L}_{\kappa\kappa}$-formulas, and suppose that every subset of it of size $< \kappa$ has a model. Since $|L_S| = \kappa$, we can enumerate Φ as $\{\phi_\alpha : \alpha < \kappa\}$. For $\lambda < \kappa$ we may thus let

$$\Phi_\lambda = \{\phi_\alpha : \alpha < \lambda\},$$

and observe that since $|\Phi_\lambda| = \lambda$, this has a model M_λ. Let $M = \prod_{\lambda<\kappa} M_\lambda/U$ be the ultraproduct of the M_λ over a κ-complete non-principal ultrafilter over κ, we obtain that for all $\alpha < \kappa$

$$\{\lambda < \kappa : \lambda > \alpha\} \in U$$

and so

$$\{\lambda < \kappa : M_\lambda \models \phi_\alpha\} \in U.$$

Consequently, Loś's Theorem for $\mathcal{L}_{\kappa\kappa}$-languages implies that M is a model for Φ. $\square$