Example Sheet #2

Examples Classes.
#1: Friday 9 February 2024, 1:30–3:30pm, **MR15**.
#2: Friday 1 March 2024, 1:30–3:30pm, **MR5**.
#3: Friday 15 March 2024, 1:30–3:30pm, **MR5**.

Presentation. Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

Marking. You can submit all of your work to Ioannis Eleftheriadis (`ie257`) as a *single pdf file* by e-mail or hand it to him on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission. Model solutions will be provided on the moodle page of the course.

(17) Assume that there is an set with a Banach measure on it and let κ be the smallest cardinality of such a set. Prove that every Banach measure on κ is κ-additive.

(18) If μ is a Banach measure on S, we say that $A \subseteq S$ is an *atom of* μ if $\mu(A) > 0$ and for each $B \subseteq A$, either $\mu(B) = \mu(A)$ or $\mu(B) = 0$. We call μ *atomless* if it does not have any atoms. Prove that if μ is atomless, then for each set $A \subseteq S$, there is some $B \subseteq A$ such that $\mu(B) = {}^1\!/_2 \cdot \mu(A)$.

(19) Assume that there is a κ-complete atomless Banach measure on κ. Prove that $\kappa \leq 2^{\aleph_0}$. Derive that if there is a real-valued measurable cardinal with an atomless Banach measure on it, then there are weakly inaccessible cardinals that are not inaccessible and CH is false.

(20) Show that if μ is a Banach measure on S that has an atom, then there is a two-valued Banach measure on S.

(21) Let U be an ultrafilter on κ. Show that U is λ-complete if and only if for each $\gamma < \lambda$ and $\{A_\alpha\,;\,\alpha < \gamma\} \subseteq U$, we have that $\bigcap_{\alpha<\gamma} A_\alpha \neq \varnothing$.

(22) Using the Axiom of Choice, show that every filter can be extended to an ultrafilter.

(23) If C is a set of subsets of Z, we say that D is the *collection generated by* C if D is minimal such that $C \subseteq D \subseteq \wp(Z)$ and D is closed under finite intersections and supersets. Let X and Y be sets, $f : X \to Y$ a function, F a filter on X and G a filter on Y. Let f_*F be the collection generated by $\{f[A]\,;\,A \in F\}$ (called the *pushout of* F) and f^*G be the collection generated by $\{f^{-1}[B]\,;\,B \in G\}$ (called the *pullback of* G).

(a) Under which conditions on f are f_*F or f^*G filters?
(b) Under which conditions on f are $\{f[A]\,;\,A \in F\}$ or $\{f^{-1}[B]\,;\,B \in G\}$ filters?
(c) If F or G are ultrafilters, are f_*F or f^*G?

(d) If F or G are κ-complete, are f_*F or f^*G?

(e) If F or G are nonprincipal, are f_*F or f^*G?

(24) Presentation Example. A cardinal κ is called an *Ulam cardinal* if there is an $\aleph_1$-complete non-principal ultrafilter on κ. Show that the smallest Ulam cardinal is a measurable cardinal.

(25) Show that the Erdős arrow notation is stable under increasing numbers on the left hand side of the arrow and decreasing numbers on the right hand side of the arrow. I.e., if $\kappa \to (\lambda)^m_\mu$ and $\kappa' \geq \kappa$, $\lambda' \leq \lambda$, $\mu' \leq \mu$, and $m' \leq m$, then $\kappa' \to (\lambda')^{m'}_{\mu'}$.

(26) Let κ be regular and $\lambda < \kappa$. Let $2^\lambda := \{f\,;\, f : \lambda \to \{0,1\}\}$ be ordered lexicographically by $f <_{\text{lex}} g$ if $f(\alpha) = 0$ and $g(\alpha) = 1$ if α is the least ordinal where f and g differ. Show that $(2^\lambda, <_{\text{lex}})$ has no strictly increasing or decreasing sequences of length κ.

(27) Let U be a ultrafilter on κ such that all elements of U have cardinality κ. Show that if U is normal, then U is κ-complete.

(28) Let κ be regular. A set $A \subseteq \kappa$ is *closed* if for each limit ordinal $\lambda < \kappa$, if $A \cap \lambda$ is cofinal in λ, then $\lambda \in A$. A set C is called a *club set* (for "closed unbounded") if it is closed and cofinal in κ. Define

$$\mathcal{C} := \{A \subseteq \kappa\,;\, \text{there is a club set } C \subseteq A\}.$$

Show that $\mathcal{C}$ is a κ-complete and normal filter on κ.

(29) Let F be a filter on a cardinal κ. Say that for $X \subseteq \kappa$, a function $f\colon X \to \kappa$ is called *regressive* if $f(\alpha) < \alpha$ for all $0 \neq \alpha \in X$. For any such function $f\colon X \to \kappa$ and $\alpha < \kappa$, write $X^f_\alpha := \{\gamma\,;\, f(\gamma) = \alpha\}$. A set S is called *F-stationary* if for all $X \in F$, we have that $X \cap S \neq \varnothing$.

Prove that the following statements are equivalent for a filter F.

(i) The filter F is closed under diagonal intersections.

(ii) For any F-stationary set S and any regressive $f\colon S \to \kappa$, there is an $\alpha < \kappa$ such that X^f_α is F-stationary.

(30) Assume that κ is measurable with a κ-complete nonprincipal ultrafilter U on κ. Use the notation of (29) and let $W := \{f\colon \kappa \to \kappa\,;\, X^f_\alpha \notin U \text{ for all } \alpha < \kappa\}$. Show that there is an $h \in W$ such that for all $f \in W$ we have that $\{\alpha\,;\, h(\alpha) \leq f(\alpha)\} \in U$. Using the notation of (23), show that g^*U is a normal κ-complete nonprincipal ultrafilter on κ.

(31) Presentation Example. Assume that κ is measurable with a κ-complete nonprincipal ultrafilter U on κ. Formulate and prove Łoś's Theorem for $\mathcal{L}_{\kappa\kappa}$-languages for the ultrapowers by U.

(32) Let S be a set of symbols for an $\mathcal{L}_{\kappa\kappa}$ language L_S. Show that if $|S| \leq \kappa$, then $|L_S| = \kappa$. Use this and (31) to give a alternative proof of the fact that every measurable cardinal is weakly compact.

[*Hint.* Use the characterisation—not proved in the course—of weak compactness via compactness of $\mathcal{L}_{\kappa\kappa}$-languages.]