Large Cardinals
Lent Term 2024
Part III of the Mathematical Tripos
University of Cambridge
Prof. Dr. B. Löwe, I. Eleftheriadis

Model Solutions for Example Sheet #1

Ioannis Eleftheriadis (`ie257`)

(1) Show for any cardinal κ that there is a definable surjection from $\mathcal{P}(x)$ to κ^+, i.e., a formula $\Phi(x, y)$ such that $F(A) = \alpha :\iff \Phi(A, \alpha)$ defines a surjective function for $A \subseteq \kappa$ and $\alpha < \kappa^+$.

[*Hint.* Consider the proof of Hartogs's Lemma; you may use that there is a bijection between κ and $\kappa \times \kappa$ for infinite κ.]

Solution. Consider the formula

$$\Phi(x, y) = y \in \mathrm{Ord} \wedge \exists R \subseteq x(R \text{ well-orders } y) \wedge \forall z \in \mathrm{Ord}(R \text{ well-orders } z \to z \subseteq y)$$

It follows that for every cardinal κ and $A \subseteq \kappa \times \kappa$ there is a unique ordinal α such that $\Phi(A, \alpha)$; since A is a well-ordering of α it follows that $\alpha \in \kappa^+$. In particular, Φ defines a function from $\mathcal{P}(\kappa \times \kappa) \to \kappa^+$. It is easy to verify that this is a surjection: for any $\alpha < \kappa^+$ there is a bijection $f : \kappa \to \alpha$, and so the relation $A := \{(x, y) \in \kappa \times \kappa : f(x) < f(y)\}$ is a well-order of κ such that $\Phi(A, \alpha)$ is true. Finally, composing Φ with the definable bijection from $\kappa \times \kappa \to \kappa$ gives a definable surjection from $\Phi(x)$ to κ^+. $\dashv$

(2) Let λ and μ be limit ordinals and $f : \mu \to \lambda$ be a function. The function f is called *cofinal in* λ if $\mathrm{ran}(f)$ is a cofinal subset of λ. Show that

$$\begin{aligned}\mathrm{cf}(\lambda) &= \min\{\mu, ; \text{there is a cofinal function with domain } \mu\} \\ &= \min\{\mu\, ; \text{there is a strictly increasing cofinal function with domain } \mu\}.\end{aligned}$$

Conclude that $\mathrm{cf}(\mathrm{cf}(\lambda)) = \mathrm{cf}(\lambda)$.

Solution. For the first equality, note that if $C \subseteq \lambda$ is cofinal, then the map $f : |C| \to \lambda$ given by the composition of a bijection between C and $|C|$ and inclusion is cofinal. Conversely, if there is $\alpha < \mathrm{cf}(\lambda)$ with $f : \alpha \to \lambda$ cofinal, then $f[a] \subseteq \lambda$ is cofinal and $|f[a]| \leq |a| < \mathrm{cf}(\lambda)$, contradiction.

The second equality follows from the fact that given $f : \mathrm{cf}(\lambda) \to \lambda$ cofinal, there is some $g : \mathrm{cf}(\lambda) \to \lambda$ strictly increasing and cofinal. Indeed, define $g : \mathrm{cf}(\lambda) \to \lambda$ by $\beta \mapsto \sup_{\delta<\beta}(f(\delta) + \beta)$. This is clearly strictly increasing and also maps into λ: if $\lambda = g(\beta)$ for some $\beta < \mathrm{cf}(\lambda)$ then $\lambda = \bigcup_{\delta<\beta}(f(\delta) + \beta)$, contradicting that $\beta < \mathrm{cf}(\lambda)$. Finally, it is easy to see that g is cofinal: if $\alpha < \lambda$ then $\exists \beta < \mathrm{cf}(\lambda)$ such that $a < f(\beta) \leq g(\beta + 1) < \lambda$.

Clearly, $\mathrm{cf}(\mathrm{cf}(\alpha)) \leq \mathrm{cf}(\alpha)$. For the other direction, pick $f : \mathrm{cf}(\mathrm{cf}(\alpha)) \to \mathrm{cf}(\alpha)$ and $g : \mathrm{cf}(\alpha) \to \alpha$ strictly increasing and cofinal. Their composition is a strictly increasing and cofinal map $\mathrm{cf}(\mathrm{cf}(\alpha)) \to \alpha$, and so $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\mathrm{cf}(\alpha))$ by the above. $\dashv$

(3) Presentation Example. Let κ be regular, η be any ordinal and $f : \kappa \to \eta$ a strictly increasing function. Define $\lambda := \bigcup \mathrm{ran}(f)$. Show that $\mathrm{cf}(\lambda) = \kappa$. Conclude that $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$ and $\mathrm{cf}(\beth_\lambda) = \mathrm{cf}(\lambda)$.

Solution. Fix some cofinal map $g : \mathrm{cf}(\lambda) \to \lambda$. Consider the map $h : \mathrm{cf}(\lambda) \to \kappa$ given by mapping $\alpha < \mathrm{cf}(\lambda)$ to the least $\beta < \kappa$ such that $g(\alpha) < f(\beta)$. This is well-defined and cofinal. Indeed, if $\gamma < \kappa$, then find some $\alpha < \mathrm{cf}(\lambda)$ with $g(\alpha) > f(\gamma)$. Since f is strictly increasing, the least β with $f(\gamma) < g(\alpha) < f(\beta)$ must be strictly greater than γ, so $h(\alpha) > \gamma$. It follows that $\mathrm{cf}(\lambda) = \kappa$ by regularity of κ.

Finally, we show that $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$. Take a strictly increasing cofinal map $\mathrm{cf}(\lambda) \to \lambda$ and compose it with $\alpha \mapsto \aleph_\alpha$. Then we have a strictly increasing cofinal map $f : \mathrm{cf}(\lambda) \to \aleph_\lambda$, and $\aleph_\lambda = \bigcup \mathrm{ran}(f)$. Since $\mathrm{cf}(\lambda)$ is regular, we use the argument from the previous part to deduce that $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$. Applying the same argument to the map $\alpha \mapsto \beth_\alpha$ it follows that $\mathrm{cf}(\beth_\lambda) = \mathrm{cf}(\lambda)$. $\dashv$

(4) Prove that every successor cardinal is regular. Note that the proof uses some fragment of the Axiom of Choice.

Solution. Fix a cardinal κ and a map $f : \kappa \to \kappa^+$. Observe that

$$|\sup_{\alpha<\kappa} f(\alpha)| \leq \kappa \times \kappa = \kappa,$$

by a standard cardinal arithmetic argument (which uses the Axiom of Choice). It follows that f cannot be cofinal, and since this holds for any such f, $\mathrm{cf}(\kappa^+) > \kappa$. Since the cofinality is a cardinal, this implies that $\mathrm{cf}(\kappa^+) = \kappa^+$, i.e. κ^+ is regular. $\dashv$

(5) A class function $F : \mathrm{Ord} \to \mathrm{Ord}$ is called a normal ordinal operation if for all $\alpha < \beta$, we have $F(\alpha) < F(\beta)$, and for all limit ordinals λ, we have $F(\lambda) = \bigcup_{\alpha<\lambda} F(\alpha)$. Prove that every normal ordinal operation has arbitrarily large fixed points, i.e., for each α, there is a $\gamma \geq \alpha$ such that $F(\gamma) = \gamma$.

Solution. Fix an ordinal α and define by recursion on ω:

$$R(0) := \alpha + 1;$$

$$R(n+1) := F(R(n)).$$

Using Replacement, let $\gamma = \bigcup_{n\in\omega} R(n)$. We argue that γ is a fixed point of F above α. First, starting with $\alpha < R(0)$ and using that F is strictly increasing, we obtain by an easy induction argument that $\gamma > \alpha$. We now distinguish two cases. If there is some $n \in \omega$ such that $R(n)$ is a fixed point of F, then by definition $\gamma = R(n)$ and so $F(\gamma) = \gamma$. On the other hand, if no $R(n)$ is a fixed point of F then the sequence $(R(n))_{n\in\omega}$ is strictly increasing, and so γ is a limit ordinal. It therefore follows by continuity of F and the fact that $(R(n))_{n\in\omega}$ is cofinal in γ that:

$$F(\gamma) = \bigcup_{\alpha<\gamma} F(\alpha) = \bigcup_{n\in\omega} F(R(n)) = \bigcup_{n\in\omega} R(n+1) = \gamma,$$

as required. $\dashv$

(6) We call the axiom system that contains all axioms of ZFC except for the Axiom of Infinity FST (for "Finite Set Theory"). Consider the property $I(\alpha)$ defined by "α is a limit ordinal and $\alpha \neq 0$". Show that the property I is a *large cardinal property* for FST in the following sense:

If FST is consistent, then FST does not prove the existence of a cardinal with property I.

Solution. Suppose that $\mathsf{FST} \vdash \exists\alpha I(\alpha)$. Since $\mathsf{FST} + \exists\alpha I(\alpha) \vdash$ Infinity, we know by modus ponens that $\mathsf{FST} \vdash$ Infinity, and hence $\mathsf{FST} \vdash \mathsf{ZFC}$. However, $\mathbf{V}_\omega$ is a model of FST that is not a model of full ZFC. Indeed, all but the Replacement scheme can be easily verified. For this, observe that if $F : \mathbf{V}_\omega \to \mathbf{V}_\omega$ is a function and $x \in \mathbf{V}_\omega$, then for all $y \in x$, $\mathrm{rank}(F(y)) < \omega$. Then $C := \{\mathrm{rank}(F(y)) : y \in x\} \subseteq \omega$ is a finite set, so $\mathrm{rank}(F[x]) \leq \sup C + 1 < \omega$. Therefore $F[x] \in \mathbf{V}_\omega$. $\dashv$

(7) Let κ be a regular cardinal. If x is any set, we write $\mathrm{tcl}(x)$ for the transitive closure of x. Define $\mathbf{H}_\kappa := \{x\,;\, |\mathrm{tcl}(x)| < \kappa\}$. Why is this a set? Which axioms of ZFC hold in $\mathbf{H}_{\aleph_1}$? Show that for any κ, $\mathbf{H}_{\kappa^+}$ cannot be a model of ZFC.

Solution. We first show that $\mathbf{H}_\kappa \subseteq \mathbf{V}_\kappa$ for all infinite ordinals κ, and hence $\mathbf{H}_\kappa$ is a set. We adapt the proof from Kenneth Kunen's Set Theory, p. 131. Let $x \in \mathbf{H}_\kappa$. We shall argue that $\mathrm{rank}(x) < \kappa$. Indeed, let $t = \mathrm{tcl}(x)$ and $S = \{\mathrm{rank}(y) : y \in t\} \subseteq \mathbf{Ord}$. Let α be the first ordinal not in S.

By definition, this implies that $\alpha \subseteq S$. If $\alpha \neq S$, let β be the least element of S larger than α, and fix some $y \in x$ with $\mathrm{rank}(y) = \beta$. By transitivity of t, $\mathrm{rank}(z) < \alpha$ for all $z \in y$, and so $\mathrm{rank}(y) = \bigcup\{\mathrm{rank}(z) + 1 : z \in y\} \leq \alpha$, contradiction. So $S = \alpha$. Therefore $|t| < \kappa \implies \alpha < \kappa$, and so $\mathrm{rank}(x) \leq \alpha < \kappa$.

We now argue that when κ is regular, $x \subseteq \mathbf{H}_\kappa$ and $|x| < \kappa$, then $x \in \mathbf{H}_\kappa$. Indeed, $\mathrm{tcl}(x) = x \cup \{\mathrm{tcl}(y) : y \in x\}$, and so $\mathrm{tcl}(x)$ is the union of $< \kappa$ sets of cardinality $< \kappa$. By regularity, $|\mathrm{tcl}(x)| < \kappa$ so $x \in \mathbf{H}_\kappa$.

With this, we show that $\mathbf{H}_\kappa \models \mathsf{ZFC} - \mathsf{Powerset}$ for all regular $\kappa > \omega$. It is easy to verify Extensionality, Foundation, Union, Pairing, and Comprehension. By the observation above, we also have that $\mathbf{H}_\kappa$ satisfies (second order) Replacement. Since $\omega \in \mathbf{H}_\kappa$, we also obtain Infinity. Finally, $\mathbf{H}_\kappa \models \forall A \exists R(R$ well orders $A)$. Indeed, being a well-order is absolute for transitive models of $\mathsf{ZF} - \mathsf{Powerset}$ (see Chapter IV, Theorem 5.4 in Kunen), and so for any $A \in \mathbf{H}_\kappa$ there is some $R \subseteq A \times A$ which is a well-order on A. By the argument above, $R \in \mathbf{H}_\kappa$ gives the required well-order in $\mathbf{H}_\kappa$. Finally, note that if κ is not a strong limit then $\mathbf{H}_\kappa \not\models$ Powerset. Indeed, if there is $\lambda < \kappa$ such that $2^\lambda \geq \kappa$ then $\mathcal{P}(\lambda)^{\mathbf{H}_\kappa} = \mathcal{P}(\lambda) \notin \mathbf{H}_\kappa$, since $|\mathcal{P}(\lambda)| = 2^\lambda$. $\dashv$

(8) Show that $\mathbf{V}_\kappa = \mathbf{H}_\kappa$ if and only if κ is inaccessible.

Solution. Assuming that κ is inaccessible, we have that $x \in \mathbf{V}_\kappa \implies x \in \mathbf{V}_\alpha$ for some $\alpha < \kappa$. By transitivity, $\mathrm{tcl}(x) \subseteq \mathbf{V}_\alpha$ and therefore $|\mathrm{tcl}(x)| \leq |\mathbf{V}_\alpha| = \alpha < \kappa$ since κ is inaccessible. Therefore $x \in \mathbf{H}_\kappa$. Conversely, suppose that $\mathbf{V}_\kappa = \mathbf{H}_\kappa$. Then if $\alpha < \kappa$, $\mathcal{P}(\alpha) \in \mathbf{V}_\kappa$ since κ is a limit ordinal, and therefore $2^\alpha \leq |\mathrm{tcl}(\mathcal{P}(\alpha))| < \kappa$. So κ is a strong limit, and therefore inaccessible. $\dashv$

(9) Suppose $M \subseteq \mathbf{V}_\lambda$ is countable and transitive. Show that the formula describing "x is a cardinal" is not absolute for M and $\mathbf{V}_\lambda$.

Solution. We know that $\mathsf{ZFC} \vdash \exists\kappa(\kappa \in \mathrm{Card} \wedge \kappa > \omega)$. Taking M to be a countable transitive submodel of $\mathbf{V}_\lambda$ it follows that there is some $\kappa \in M$ such that $M \models \kappa \in \mathrm{Card} \wedge \kappa > \omega$. Since M is transitive it follows that κ is a countable ordinal above ω, and so in particular there is (in $\mathbf{V}_\lambda$) a bijection to ω, i.e. κ is not a real cardinal. $\dashv$

(10) Show that every worldly cardinal is an aleph fixed point.

Solution. Let κ be worldly. We first argue that $\phi(x, y) =$ "$x, y \in \mathrm{Ord} \wedge y$ is the least such that there is no surjection $x \to y$" is absolute between $\mathbf{V}_\kappa$ and $\mathbf{V}$. Firstly, since this statement is Π_1 it is downwards absolute. So, consider $\alpha < \beta \in \mathrm{Ord} \cap \mathbf{V}_\kappa$ such that $\phi(\alpha, \beta)$ is false. Then, either there is some $\gamma < \beta$ such that there is no surjection from $\alpha \to \gamma$ or there is a surjection $\alpha \to \beta$. Clearly, since $\gamma \in \mathrm{Ord} \cap \mathbf{V}_\kappa$ the former cannot possibly be true. On the other hand if $f : \alpha \to \beta$ is any map, then $f \subseteq \alpha \times \beta \in \mathbf{V}_{\beta+3}$, and so $f \in \mathbf{V}_{\beta+4}$. It follows that $f \in \mathbf{V}_\kappa$ and since being a surjection is absolute between transitive models, $\mathbf{V}_\kappa \models \neg\phi(\alpha, \beta)$. From this we deduce that for all cardinals $\mu < \kappa$, $(\mu^+)^{\mathbf{V}_\kappa} = \mu^+$.

Hence, we argue that for all $\lambda \in \mathbf{V}_\kappa$, $(\aleph_\lambda)^{\mathbf{V}_\kappa} = \aleph_\lambda$. Proceed by induction. Since $\mathbf{V}_\kappa \models \mathsf{ZFC}$, $(\omega)^{\mathbf{V}_\kappa} = \omega$, and so $(\aleph_0)^{\mathbf{V}_\kappa} = \aleph_0$. Assume that $(\aleph_\alpha)^{\mathbf{V}_\kappa} = \aleph_\alpha$. Then $(\aleph_{\alpha+1})^{\mathbf{V}_\kappa} = (\aleph_\alpha^+)^{\mathbf{V}_\kappa} = \aleph_{\alpha+1}$. Finally the limit case follows from absoluteness of unions.

So, assume for a contradiction that κ is not an aleph fixed point. In particular, this implies that $\kappa = \aleph_\lambda$ for some cardinal $\lambda < \kappa$. By the above claim, $(\aleph_\lambda)^{\mathbf{V}_\kappa} = \aleph_\lambda \in \mathbf{V}_\kappa$, and so $\kappa \in \mathbf{V}_\kappa$; contradiction. $\dashv$

(11) If T is any theory, we write $T^\star := T + \mathsf{Cons}(T)$. Define by recursion

$$\mathsf{ZFC}(0) := \mathsf{ZFC} \text{ and}$$
$$\mathsf{ZFC}(n+1) := (\mathsf{ZFC}^{(n)})^\star.$$

We write WorC for "there is a worldly cardinal". Show that WorC implies $\mathsf{ZFC}^{(n)}$ for all $n \in \omega$.

Solution. Assume WorC and let κ be a worldly cardinal. Clearly $\mathbf{V} \models \mathsf{ZFC}$, and since κ is worldly, $\mathbf{V}_\kappa \models \mathsf{ZFC}$. So, assume that $\mathbf{V} \models \mathsf{ZFC}(n)$ and $\mathbf{V}_\kappa \models \mathsf{ZFC}(n)$. It follows that $\mathbf{V} \models$ "$\mathbf{V}_\kappa \models \mathsf{ZFC}(n)$"

and so $\mathbf{V} \models \mathsf{Cons}(\mathsf{ZFC}(n))$, implying that $\mathbf{V} \models \mathsf{ZFC}(n+1)$. Moreover since arithmetic statement are absolute between transitive models we obtain that $\mathbf{V}_\kappa \models \mathsf{Cons}(\mathsf{ZFC}(n))$, and so $\mathbf{V}_\kappa \models \mathsf{ZFC}(n+1)$. Proceeding by induction, we get that $\mathbf{V} \models \mathsf{ZFC}(n)$ for all $n \in \omega$. $\dashv$

(12) Let β be any ordinal and $R \subseteq \mathbf{V}_\beta$. An ordinal $\alpha < \beta$ is called an *R-Lévy ordinal for β* if $(\mathbf{V}_\alpha, \in, R \cap \mathbf{V}_\alpha)$ is an elementary substructure of $(\mathbf{V}_\beta, \in, R)$. Show that no α can be an R-Lévy ordinal for all $R \subseteq \mathbf{V}_\beta$.

Solution. Take $R = \mathbf{V}_\alpha$, and suppose that $(\mathbf{V}_\alpha, \in, \mathbf{V}_\alpha) \preccurlyeq (\mathbf{V}_\beta, \in, \mathbf{V}_\alpha)$. Since $\mathbf{V}_\beta \models \exists x(\neg R(x))$, then so does $\mathbf{V}_\alpha$. But this is a contradiction. $\dashv$

(13) Presentation Example. Show the following theorem due to Lévy: an ordinal κ is an inaccessible cardinal if and only if for each $R \subseteq \mathbf{V}_\kappa$ there is an R-Lévy ordinal for κ.

Solution. Suppose that κ is inaccessible, and let $R \subseteq \mathbf{V}_\kappa$. Define by recursion on ω: $\alpha_0 = \emptyset, \alpha_{n+1} =$ the least $\beta \geq \alpha_n$ such that whenever $y_1, \dots, y_k \in \mathbf{V}_{\alpha_n}$ and $(\mathbf{V}_\kappa, \in, R) \models \exists x \phi(x, y_1, \dots, y_k)$ for some formula ϕ, there is an $x_0 \in \mathbf{V}_\beta$ such that $(\mathbf{V}_\kappa, \in, R) \models \phi(x_0, y_1, \dots, y_k)$. Since κ is inaccessible, $|V_{\alpha_n}| < \kappa$ and so $\alpha_{n+1} < \kappa$. Finally take $\alpha = \bigcup_\omega \alpha_n$. Using Tarski-Vaught, we may easily verify that $(\mathbf{V}_\alpha, \in, R \cap \mathbf{V}_\alpha) \preccurlyeq (\mathbf{V}_\kappa, \in, R)$. Note that by starting with any arbitrary $\alpha_0 = \lambda < \kappa$, the above argument shows that $\{\alpha : (\mathbf{V}_\alpha, \in, R \cap \mathbf{V}_\alpha) \preccurlyeq (\mathbf{V}_\kappa, \in, R)\}$ is in fact unbounded in κ.

For the converse, notice first that κ must necessarily be infinite. If κ is not regular, then there is $\beta < \kappa$ and $f : \beta \to \kappa$ cofinal. Let $R = \{\beta\} \cup f$ and find $\alpha < \kappa$ such that $(\mathbf{V}_\alpha, \in, R \cap \mathbf{V}_\alpha) \preccurlyeq (\mathbf{V}_\kappa, \in, R)$. Since β is the only ordinal in R, we see that $\beta \in \mathbf{V}_\alpha$ by elementarity. But then there is some $\gamma < \beta$ in $\mathbf{V}_\alpha$ with $\alpha < f(\gamma) < \kappa$ and $f(\gamma) \in \mathbf{V}_\alpha$, contradiction.

Also, if κ is not a strong limit then we can find $\beta < \kappa$ with $2^\beta \geq \kappa$. Find a surjection $g : \mathcal{P}(\beta) \to \kappa$ and take $R = \{\beta + 1\} \cup g$. By assumption, there is $\alpha < \kappa$ such that $(\mathbf{V}_\alpha, \in, R \cap \mathbf{V}_\alpha) \preccurlyeq (\mathbf{V}_\kappa, \in, R)$. Since $\beta + 1 \in V_\alpha$, it follows that $\mathcal{P}(\beta) \in \mathbf{V}_\alpha$ and so again we can find some $x \in \mathcal{P}(\beta)$ such that $g(x) = \alpha \in \mathbf{V}_\alpha$, contradiction. $\dashv$

(14) Let $2\mathsf{IC}$ be the statement "there are $\lambda < \kappa$ such that both λ and κ are inaccessible". Show that if $\mathsf{ZFC} + \mathsf{IC}$ is consistent, then IC does not imply $2\mathsf{IC}$.

Solution. Assume for a contradiction that $V \models \mathsf{ZFC} + \mathsf{IC}$ and $\mathsf{ZFC} + \mathsf{IC} \vdash 2\mathsf{IC}$. It follows that $V \models 2\mathsf{IC}$, so consider the first two inaccessibles $\lambda < \kappa$. It follows that $\mathbf{V}_\kappa \models \mathsf{IC}$ by downwards absoluteness of inaccessibility. By our assumption this implies that $\mathbf{V}_\kappa \models 2\mathsf{IC}$, and so there are $\lambda_1 < \lambda_2 < \kappa$ such that $\mathbf{V}_\kappa \models$ "λ_i is inaccessible". Since κ is inaccessible itself, this implies that λ_1 and λ_2 are indeed inaccessible, contradicting that there is only one inaccessible below κ. $\dashv$

(15) Prove that under appropriate consistency assumptions, the formula describing "λ is inaccessible" is not absolute for transitive models of ZFC. Comment on the consistency assumptions: what are they and why are they needed?

Solution. Assume $2\mathsf{IC}$, and let $\kappa < \lambda$ be inaccessibles. It follows that $\mathbf{V}_\lambda \models \mathsf{ZFC} + \mathsf{IC}$. Take a countable elementary submodel of $\mathbf{V}_\lambda$ and consider its Mostowski collapse M. This is now a countable transitive model of $\mathsf{ZFC} + \mathsf{IC}$, so there is some $\alpha \in M$ such that $M \models$ "α is inaccessible". However, transitivity of M implies that α is in reality a countable ordinal, and therefore not inaccessible. $\dashv$

(16) Let $\infty\mathsf{IC}$ be the statement "for any ordinal α, there is an inaccessible cardinal $\kappa > \alpha$". Assume $\infty\mathsf{IC}$ and consider the ordinal operation $\iota : \mathrm{Ord} \to \mathrm{Ord}$ such that $\iota(\alpha)$ is the αth inaccessible cardinal. Show that ι is not a normal ordinal operation and that if $\mathsf{ZFC} + \infty\mathsf{IC}$ is consistent, it cannot prove that ι has any fixed points.

Solution. By definition, $\iota(\omega)$ is inaccessible and therefore regular. It follows that $\mathrm{cf}(\iota(\omega)) \neq \omega$ and therefore $\iota(\omega) \neq \cup_{n<\omega} \iota(n)$, i.e. ι is not continuous. Write FP_ι for the statement that ι has a fixed point, and assume for a contradiction that $\mathsf{ZFC} + \infty\mathsf{IC}$ is consistent and $\mathsf{ZFC} + \infty\mathsf{IC} \vdash \mathsf{FP}_\iota$. Let κ be the least fixed point of ι. It follows that any ordinal $\alpha < \kappa$ is not a fixed point of ι, and so $\alpha < \iota(\alpha) < \iota(\kappa) = \kappa$.

In particular, for any $\alpha < \kappa$ there is an inaccessible above α in $\mathbf{V}_\kappa$, i.e. $\mathbf{V}_\kappa \models \infty\mathsf{IC}$. Our assumption implies that $\mathbf{V}_\kappa \models \mathsf{FP}_\iota$, and therefore there some $\lambda < \kappa$ which is a fixed point of ι, contradicting that κ is the least such fixed point. $\dashv$