Large Cardinals
Lent Term 2024
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #1

Examples Classes.
#1: Friday 9 February 2024, 1:30–3:30pm, **MR15**.
#2: Friday 1 March 2024, 1:30–3:30pm, **MR5**.
#3: Friday 15 March 2024, 1:30–3:30pm, **MR5**.

Presentation. Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

Marking. You can submit all of your work to Ioannis Eleftheriadis (`ie257`) as a *single pdf file* by e-mail or hand it to him on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission. Model solutions will be provided on the moodle page of the course.

(1) Show for any cardinal κ that there is a definable surjection from $\wp(\kappa)$ to κ^+, i.e., a formula $\Phi(x,y)$ such that $F(A) = \alpha :\iff \Phi(A,\alpha)$ defines a surjective function for $A \subseteq \kappa$ and $\alpha < \kappa^+$.

[*Hint.* Consider the proof of Hartogs's Lemma; you may use that there is a bijection between κ and $\kappa \times \kappa$ for infinite κ.]

(2) Let λ and μ be limit ordinals and $f\colon \mu \to \lambda$ be a function. The function f is called *cofinal in* λ if $\mathrm{ran}(f)$ is a cofinal subset of λ. Show that

$$\begin{aligned}\mathrm{cf}(\lambda) &= \min\{\mu, ; \text{there is a cofinal function with domain } \mu\} \\ &= \min\{\mu\,; \text{there is a strictly increasing cofinal function with domain } \mu\}.\end{aligned}$$

Conclude that $\mathrm{cf}(\mathrm{cf}(\lambda)) = \mathrm{cf}(\lambda)$.

(3) Presentation Example. Let κ be regular, η be any ordinal and $f\colon \kappa \to \eta$ a strictly increasing function. Define $\lambda := \bigcup \mathrm{ran}(f)$. Show that $\mathrm{cf}(\lambda) = \kappa$. Conclude that $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$ and $\mathrm{cf}(\beth_\lambda) = \mathrm{cf}(\lambda)$.

(4) Prove that every successor cardinal is regular. Note that the proof uses some fragment of the Axiom of Choice.

(5) A class function $F\colon \mathrm{Ord} \to \mathrm{Ord}$ is called a *normal ordinal operation* if for all $\alpha < \beta$, we have $F(\alpha) < F(\beta)$, and for all limit ordinals λ, we have $F(\lambda) = \bigcup_{\alpha<\lambda} F(\alpha)$. Prove that every normal ordinal operation has arbitrarily large fixed points, i.e., for each α, there is a $\gamma \geq \alpha$ such that $F(\gamma) = \gamma$.

(6) We call the axiom system that contains all axioms of ZFC except for the Axiom of Infinity FST (for "Finite Set Theory"). Consider the property $\mathsf{Inf}(\alpha)$ defined by "α is a limit ordinal and $\alpha \neq 0$". Show that InfC is a *large cardinal property* for FST in the following sense:

If FST is consistent, then FST does not prove InfC.

(7) Let κ be a regular cardinal. If x is any set, we write $\mathrm{tcl}(x)$ for the transitive closure of x. Define $\mathbf{H}_\kappa := \{x\,;\, |\mathrm{tcl}(x)| < \kappa\}$. Prove that $\mathbf{H}_\kappa$ is a set. Which axioms of ZFC hold in $\mathbf{H}_{\aleph_1}$? Show that for any κ, $\mathbf{H}_{\kappa^+}$ cannot be a model of ZFC.

(8) Show that $\mathbf{V}_\kappa = \mathbf{H}_\kappa$ if and only if κ is inaccessible.

(9) Suppose $M \subseteq \mathbf{V}_\lambda$ is countable and transitive. Show that the formula describing "x is a cardinal" is absolute for M and $\mathbf{V}_\lambda$.

(10) Show that every worldly cardinal is an aleph fixed point.

(11) If T is any theory, we write $T^* := T + \mathrm{Cons}(T)$. Define by recursion

$$\mathsf{ZFC}^{(0)} := \mathsf{ZFC} \text{ and}$$
$$\mathsf{ZFC}^{(n+1)} := (\mathsf{ZFC}^{(n)})^*.$$

We write WorC for "there is a worldly cardinal". Show that WorC implies $\mathsf{ZFC}^{(n)}$ for all $n \in \omega$.

(12) Let β be any ordinal and $R \subseteq \mathbf{V}_\beta$. An ordinal $\alpha < \beta$ is called an *R-Lévy ordinal for β* if $(\mathbf{V}_\alpha, \in, R \cap \mathbf{V}_\alpha)$ is an elementary substructure of $(\mathbf{V}_\beta, \in, R)$. Show that no α can be an R-Lévy ordinal for all $R \subseteq \mathbf{V}_\beta$.

(13) Presentation Example. Show the following theorem due to Lévy: an ordinal κ is an inaccessible cardinal if and only if for each $R \subseteq \mathbf{V}_\kappa$ there is an R-Lévy ordinal for κ.

(14) Let $2\mathsf{IC}$ be the statement "there are $\kappa < \lambda$ that are both inaccessible". Show that if IC is consistent, it cannot prove $2\mathsf{IC}$.

(15) Prove that under appropriate consistency assumptions, the formula describing "λ is inaccessible" is not absolute for transitive models of ZFC. Comment on the consistency assumptions: what are they and why are they needed?

(16) Let $\infty\mathsf{IC}$ be the statement "for any ordinal α, there is an inaccessible cardinal $\kappa > \alpha$". Assume $\infty\mathsf{IC}$ and consider the ordinal operation $\iota\colon \mathrm{Ord} \to \mathrm{Ord}$ such that $\iota(\alpha)$ is the αth inaccessible cardinal. Show that ι is not a normal ordinal operation and that if $\mathsf{ZFC} + \infty\mathsf{IC}$ is consistent, it cannot prove that ι has any fixed points.