XVI

LECTIO ULTIMA
SIXTEENTH LECTURE

LARGE CARDINALS

15 March 2023

KUNEN'S INCONSISTENCY

From Lecture XV

Proof. Suppose $j: V \longrightarrow M$ is elementary with $\mathrm{crit}(j) = \kappa$.

Then the least fixed pt of j above κ is

$$\kappa_0 := \kappa$$
$$\kappa_{i+1} := j(\kappa_i)$$
$$\hat{\kappa} := \sup\{\kappa_i \,;\, i \in \omega\}$$

By the usual arguments, $j(\hat{\kappa}) = \hat{\kappa}$.

Kunen's Lemma. In this situation $\{j(\alpha) \,;\, \alpha \in \hat{\kappa}\} \notin M$.

Remark. In the proof of Kunen's lemma, we didn't need that $V \models ZFC$. We only used ZFC to apply the Erdős–Hajnal result that there is an ω-Jónsson fn for $\hat{\kappa}$.

Thus if V is just large enough to contain that ω-Jónsson fn for $\hat{\kappa}$, the proof still goes through.

An ω-Jónsson fn $f: [\hat{\kappa}]^\omega \longrightarrow \hat{\kappa}$ lives in $V_{\hat{\kappa}+2}$.

<u>Corollary</u> If $j: V_{\delta+2} \longrightarrow M$ is an elementary embedding with $\mathrm{crit}(j) = \kappa$ and

$$\delta \geq \hat{\kappa}$$

where $\hat{\kappa}$ is the least j-fixed pt.

Then $M \neq V_{\delta+2}$.

<u>Corollary</u> (ZFC) There is no nontrivial elementary embedding

$$j: V_{\delta+2} \longrightarrow V_{\delta+2}$$

for any δ.

[If $\kappa := \mathrm{crit}(j) < \delta$, then by induction $\kappa_i < \delta$, and so $\hat{\kappa} \leq \delta$. Then the claim follows from previous Cor.]

Maxim called "ONE STEP BACK FROM DESASTER".
leads to slightly weaker statements:

I1 There is nontrivial elem. emb.

$$j: V_{\delta+1} \longrightarrow V_{\delta+1}$$

some δ.

I for "inconsistency"

$\longrightarrow$ I3 There is nontrivial elem. emb

$$j: V_{\delta} \longrightarrow V_{\delta}$$

some δ.

I2 is something between I1 & I3

Clearly, I.1 implies I3.

The axioms were thought to be inconsistent.

__Note__ These axioms have an algebraic character since the embeddings become operations on V_δ or $V_{\delta+1}$.

(13)

Two operations: if i, j are emb. on V_δ, then $i \circ j$ is the concatenation

ij is "i applied to j as set-theoretical object"

These operations satisfy:

$$i(j \circ k) = ij \circ ik$$

$$i \circ j = ij \circ i$$

$$i(jk) = ij(ik)$$

LEFT DISTRIBUTIVE LAW.

[Related to __braid groups__!]

I1 & I3 were not proposed as Large Cardinal Axioms, but as technical variants of Kunen's inconsistency; they were not expected to be consistent or of any mathematical interest.

Kanamori, THE HIGHER INFINITE, pp. 328–329

Attitudes about and expectations concerning I1-I3 have evolved since their formulation, from skepticism toward confidence and acceptance. In Solovay-Reinhardt-Kanamori [78: 109] we had written: "It seems likely that I1, I2, and I3 are all inconsistent since they appear to differ from the proposition proved inconsistent by Kunen only in an inessential technical way." However, this may not be the case. As mentioned earlier, the Axiom of Choice figures prominently in the proof of Kunen's result by providing a well-ordering of $\mathcal{P}(\lambda)$, and such well-orderings first appear, in coded form, in $V_{\lambda+2} - V_{\lambda+1}$. Hence, any refutations of I1-I3 would have to have a different basis. Also, recent work has provided some extrinsic evidence for their coherence and consistency:

From Example Sheet #3:

(33) Presentation Example. Let μ be a cardinal. We say that a set $X \subseteq \mathbf{V}_\lambda$ is *closed under μ-sequences* if $X^\mu \subseteq X$. Show that M is closed under κ-sequences, but not under κ^+-sequences.

[*Hint.* For the second claim, show that $s : \kappa^+ \to M : \alpha \mapsto j(\alpha)$ is not an element of M.]

<u>Definition</u> A cardinal κ is called <u>α-supercompact</u> if there is an embedding j that is α-supercompact.

Embedding j is called α-supercompact if $j : V \to M$, $\mathrm{crit}(j) = \kappa$ & $M^\alpha \subseteq M$.

[i.e., if $f : \alpha \to M$, then $f \in M$.]

So, example (33) says:

Every embedding w/ $\mathrm{crit}(j) = \kappa$ is κ-supercompact;

the ultrapower embedding is not κ^+-supercompact.

Def. κ is supercompact if it's α-sc. for all α.

Similarly to strength, the supercompact analogue of a Reinhardt cardinal would be a cardinal κ with embedding j that is α-sc. for all α.

Kunen's Lemma implies that no embedding can be $\hat{\kappa}$-sc. where $\hat{\kappa}$ is its first fixed pt bigger than crit (j).

So, sc. analogues of Reinhardt cardinals don't exist either.

CONSISTENCY STRENGTH ISN'T ALWAYS ABOUT SIZE

Φ cardinal property

$$\iota_\Phi := \begin{cases} 0 & \text{if } \neg \Phi C \\ \kappa & \text{if } \kappa \text{ is the least s.t. } \Phi(\kappa) \end{cases}$$

least Φ-cardinal

$$\Phi <_R \Psi \quad :\Longleftrightarrow \quad \mathrm{ZFC} \vdash \iota_\Phi = \iota_\Psi = 0 \ \vee \ \iota_\Phi < \iota_\Psi .$$

"REFLECTS"

Clearly $\Phi <_R \Phi$ means ΦC is inconsistent.

Our reflection arguments have shown:

$$\boxed{I <_R W <_R M <_R O_1 <_R S_2}$$

$O_k(\kappa) :\Longleftrightarrow \kappa$ has Mitchell order $\geq k$

$o(\kappa) \geq k$

$S_\alpha(\kappa) :\Longleftrightarrow \kappa$ is α-strong

It is tempting to expect that if

$$\Phi C <_{\text{Cons}} \Psi C,$$

then $\quad \Phi <_R \Psi.$

But that is not true.

Let's analyse this:

1. $<_R$ depends too much on details.

   E.g. $\quad IC <_{\text{Cons}} 2IC <_{\text{Cons}} 3IC.$

We could define

$$2I^-(\kappa) :\Longleftrightarrow \exists\lambda\,(\kappa<\lambda \wedge I(\kappa)\wedge I(\lambda))$$
$$2I^+(\kappa) :\Longleftrightarrow \exists\lambda\,(\lambda<\kappa \wedge I(\kappa)\wedge I(\lambda))$$

Clearly $\quad 2I^+C \Longleftrightarrow 2I^-C$

therefore $\quad 2I^+C \equiv_{\text{Cons}} 2I^-C.$

But $\quad \iota_{2I^-} <_R \iota_{2I^+}.$

(2) Things are not necessarily comparable with $<_R$:

$$\Phi(\kappa) := I(\kappa) \wedge (WC \Longrightarrow W(\kappa))$$

$$\Phi C \Longleftrightarrow IC$$

Thus $\Phi C \equiv_{Cons} IC$.

But $I \not<_R \Phi$ [in models of $IC \wedge \neg WC$]

and $\Phi \not<_R I$ [in models of WC].

(3) $<_R$ does not always reflect consistency strength.

$$\Phi(\kappa) :\Longleftrightarrow WC \wedge I(\kappa).$$

$$\Phi C \Longleftrightarrow WC$$

so $\Phi C \equiv_{Cons} WC$,

but if ΦC holds, then $\underset{\kappa_I}{\underset{\shortparallel}{\kappa_\Phi}} <_R \kappa_W$.

Examples (1) to (3) are all about technical failing with properties that could be seen as "unnatural" (playing around with propositional logic). But phenomena like this can happen with <u>proper</u> LCA.

Annals of Mathematical Logic 10 (1976) 33 - 57

HOW LARGE IS THE FIRST STRONGLY COMPACT CARDINAL?
or
A STUDY ON IDENTITY CRISES

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It is proved that if strongly compact cardinals are consistent, then it is consistent that the first such cardinal is the first measurable. On the other hand, if it is consistent to assume the existence of supercompact cardinal, then it is consistent to assume that it is the first strongly compact cardinal.

Menachem MAGIDOR

IDENTITY CRISES

We say an identity crisis is a situation where we have two LC notions Φ and Ψ with

$$\Phi C <_{Cons} \Psi C$$

and it is consistent to have

$$\Phi C \wedge \Psi C \wedge L_{\Phi} = L_{\Psi}.$$

Magidor's example: $St(\kappa) :\Longleftrightarrow \kappa$ is strongly compact

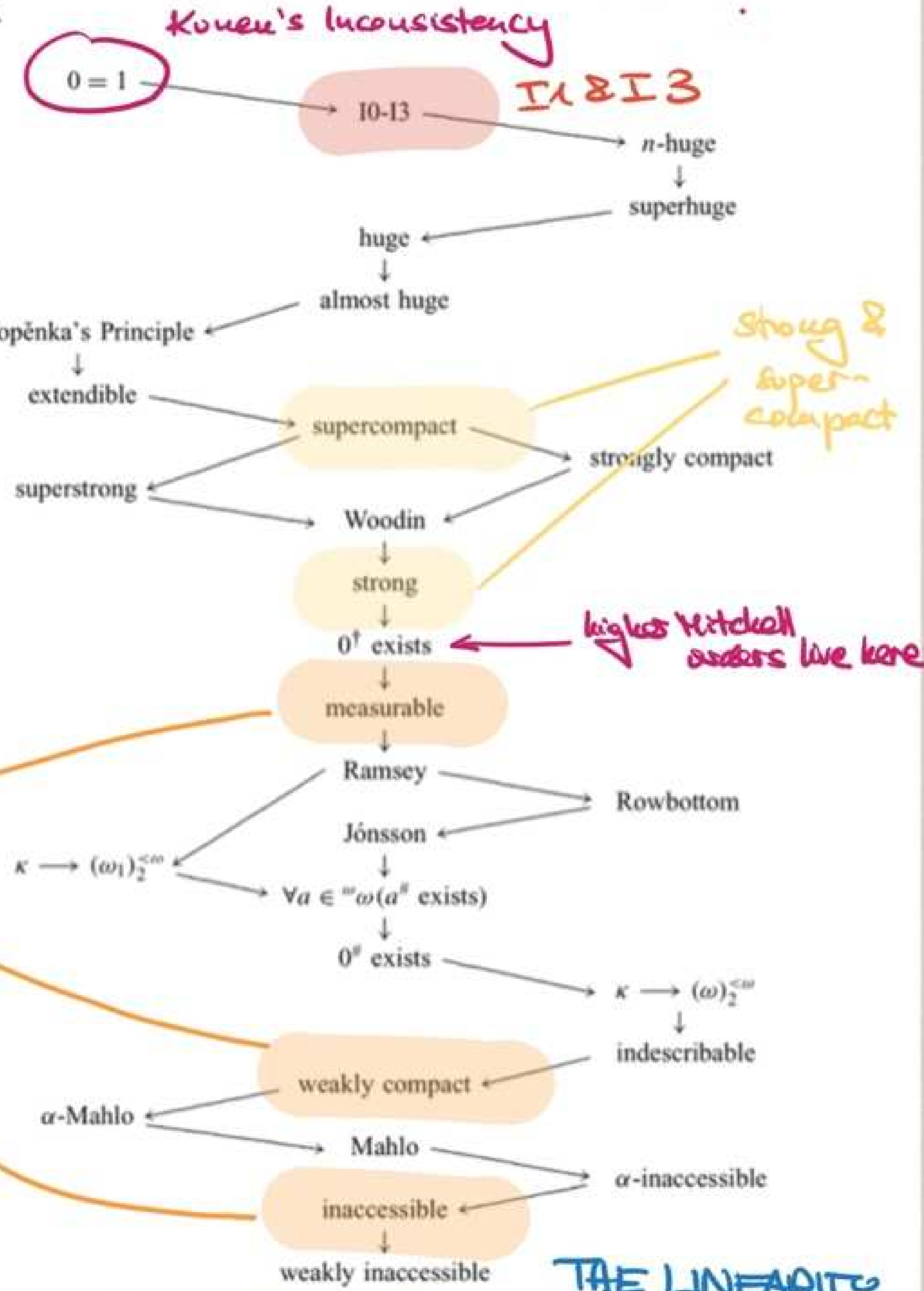
<u>Consistent</u>: $MC \wedge StC \wedge L_M = L_{St}.$

Kanamori,

THE HIGHER INFINITE

Chart of Cardinals

The arrows indicates direct implications or relative consistency implications, often both.



M, W, I
The protagonists of this lecture course

The hierarchy is largely linear!

THE LINEARITY PHENOMENON

REVISION CLASS
in Easter term:

Friday 19 MAY 2023
$3^{30}-5^{30}$ MR 9