XV Large Cardinals

PENULTIMATE (= Fifteenth) Lecture

13 March 2023

Φ <u>α-stable</u> if Φ absolute for transitive models M s.t. $V_{\kappa+\alpha} \subseteq M$

j <u>α-strong</u> if $V_{\kappa+\alpha} \subseteq M$

<u>Thus</u> If j is α-strong with $\mathrm{crit}(j) = \kappa$, then α-stable properties reflect at κ.

<u>Example</u> "measurable" $\longrightarrow$ 2-stable
"inaccessible" $\longrightarrow$ 1-stable

<u>Corollary</u> If j is 2-strong, then measurability reflects.

$IC <_{Cons} WC <_{Cons} MC <_{Cons}$ SURVIVING CARDINALS EXIST

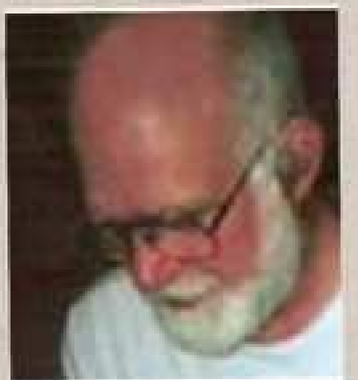
Rewrite "surviving" as a combinatorial property of ultrafilters:

Let U, U' be normal, κ-complete, nonp. ultrafilters on κ

$$U \lhd U' \qquad \text{"survives"}$$

iff $U \in M$ where M is the Mostowski collapse of V^κ / U'.

Also known as Mitchell order.



Bill Mitchell

MITCHELL LEMMA (Mitchell characterisation) TFAE:

(i) $U \lhd U'$

(ii) there is $X \in U'$ and $\{U_\alpha ; \alpha \in X\}$ s.t. U_α is normal α-complete nonp. uf. on α for all $\alpha \in X$ and

$$Z \in U \Longleftrightarrow \{\alpha ; Z \cap \alpha \in U_\alpha\} \in U'.$$

Proof. Observe that by normality $[(\mathrm{id}) = \kappa]$, we have that for $Z \subseteq \kappa$, $f_Z(\alpha) := Z \cap \alpha$ represents Z, i.e., $(f_Z) = Z$.

Prove (i) $\Longrightarrow$ (ii).

Assume $U \lhd U'$, so $U \in M$ where $M \cong V^\kappa/U'$.

So there is some $\underset{\sim}{g}$ s.t. $U = [\underset{\sim}{g}]$.

In particular, $M \models [\underset{\sim}{g}]$ is a normal nonpr. $[id]$-complete uf on $[id]$

$$\overset{\text{Łoś}}{\Longleftrightarrow} X := \{\alpha;\ \underset{\sim}{g}(\alpha) \text{ is a normal nonpr. } id(\alpha)\text{-complete uf on } id(\alpha)\} \in U'.$$

So define $U_\alpha := \underset{\sim}{g}(\alpha)$.

Now check that $Z \in U \Longleftrightarrow \{\alpha;\ Z \cap \alpha \in U_\alpha\} \in U'$

$$Z \in U \Longleftrightarrow [f_Z] \in [\underset{\sim}{g}]$$
$$\overset{\text{Łoś}}{\Longleftrightarrow} \{\alpha;\ f_Z(\alpha) \in \underset{\sim}{g}(\alpha)\} \in U'$$
$$\Longleftrightarrow \{\alpha;\ Z \cap \alpha \in U_\alpha\} \in U'.$$

(ii) $\Longrightarrow$ (i). Just reverse: by assumption have X, U_α. Define

$$\underset{\sim}{g}(\alpha) := U_\alpha.$$

Then $[\underset{\sim}{g}] = U$. q.e.d.

Corollary The existence of U, U' s.t. (*) $U \lhd U'$ is a 2-stable property.

Theorem (Mitchell 1974)

$\lhd$ is irreflexive, transitive, & wellfounded.

[No proof. Irreflexivity is known from earlier; transitivity is just a simple application of the Mitchell Lemma.]

Def. $o(U)$ is the rank of U in $\lhd$.

MITCHELL ORDER OF U

$o(\kappa) := \{o(U) ; U$ is a normal uf. κ-compl. uf on $\kappa\}$

Corollary If $o(\kappa) \geq 1$, then κ is surviving.

Note that (*) implies that $o(\kappa) \geq 1$ is 2-stable.

Theorem If κ is 2-strong, then the property $o(\kappa) \geq 1$ reflects at κ. Thus the least 2-strong cardinal has κ many surviving cardinals below it.

More on strength

Had before:

κ α-strong $:\iff \exists j\ j: V \to M$ i.s. α-strong

If you don't like the class quantifier V_λ

Definition A cardinal κ is called STRONG if it is α-strong for all α.

Note that this notion cannot have a single witness object since otherwise there is some embedding strong enough so that STRENGTH reflects below κ.

$$\forall\alpha\,\exists j_\alpha : V \to M \quad \alpha\text{-strong}$$

Even stronger

Definition A cardinal κ is called Reinhardt if there is an embedding

$$j: V \to M \text{ s.t.}$$

$\mathrm{crit}(j) = \kappa$

for all α $V_\alpha \subseteq M$

[equivalently $M = V$].

Theorem (ZFC, Kunen)

There are no Reinhardt cardinals.

Remark This proof uses AC in a nontrivial way. In particular, it is open whether Kunen's theorem can be proved in ZF.

Ken Kunen 1943-2020

KUNEN'S INCONSISTENCY

Proof. Suppose $j: V \longrightarrow M$ is elementary with $\mathrm{crit}(j) = \kappa$.

Then the least fixed pt of j above κ is

$$\kappa_0 := \kappa$$
$$\kappa_{i+1} := j(\kappa_i)$$
$$\hat{\kappa} := \sup\{\kappa_i \,;\, i \in \omega\}$$

By the usual arguments, $j(\hat{\kappa}) = \hat{\kappa}$.

Kunen's Lemma In this situation $\{j(\alpha) \,;\, \alpha \in \hat{\kappa}\} \notin M$.

Corollary $M \neq V$, so Kunen's Inconsistency follows.

Proof of Kunen's Lemma

Definition Let μ be a cardinal, then a function $f : [\mu]^{\omega} \longrightarrow \mu$ is called ω-Jónsson for μ if whenever $X \subseteq \mu$ has size μ, then

$$\{f(Z) ; Z \in [X]^{\omega}\} = \mu.$$

Bjarni Jónsson 1920 - 2016

Theorem (Erdős-Hajnal 1966) (ZFC) For every μ, there is an ω-Jónsson function for μ.

[Proof not terribly difficult, cf. Kanamori Theorem 23.13.]

So, in our situation, we have an ω-Jónsson function f for $\hat{\kappa}$. [since $\hat{\kappa}$ is j-fixed pt]

$V \models f$ is ω-Jónsson for $\hat{\kappa}$.

$M \models j(f)$ is ω-Jónsson for $j(\hat{\kappa}) = \hat{\kappa}$.

Note $X := \{j(\alpha) ; \alpha \in \hat{\kappa}\}$ is a subset of $\hat{\kappa}$ of size $\hat{\kappa}$

Suppose towards a contradiction that $X \in M$.

Then since $M \models j(f)$ is ω-Jónsson:

$$\{j(f)(Z);\ Z \in [X]^\omega\} = \hat{\kappa}.$$

[Definition of $j(f)$ is ω-Jónsson.]

$X := \{j(\alpha);\ \alpha \in \hat{\kappa}\}$

Thus if $Z \in [X]^\omega$ is fixed, there is $Y \in [\hat{\kappa}]^\omega$ s.t. $Z = j(Y)$.

[In general, if $f(a) = b$, then $j(f)(j(a)) = j(b)$.]

$$j(f)(Z) = j(f)(j(Y)) = j(f(Y)) \in \{j(\alpha);\ \alpha \in \hat{\kappa}\}.$$

But this means

$$\hat{\kappa} = \{j(f)(Z);\ Z \in [X]^\omega\} \subseteq \{j(\alpha);\ \alpha \in \hat{\kappa}\}.$$

But $\kappa \notin \{j(\alpha);\ \alpha \in \hat{\kappa}\}$.

Contradiction!

So $\{j(\alpha);\ \alpha \in \hat{\kappa}\} \notin M$. q.e.d.