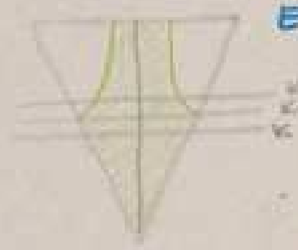
XIII

Thirteenth Lecture of LARGE CARDINALS
6 March 2023

DEFINITION κ is an EMBEDDING CARDINAL



$E(\kappa) :\iff \exists\lambda\exists M\exists j$
$\kappa<\lambda$ & λ is inaccessible
& $M\subseteq V_\lambda$ & $j:V_\lambda\to M$
& M is transitive &
j is elementary &
$\kappa = \mathrm{crit}(j)$

$E(\kappa, M) :\iff M$ witnesses $E(\kappa)$
$MI(\kappa) :\iff \exists\lambda\ \kappa<\lambda$ & $I(\lambda)$ & $M(\kappa)$

From Lecture XII

FUNDAMENTAL THEOREM ON MEASURABLE CARDINALS

Theorem Let κ be an uncountable cardinal.
TFAE: (i) $E(\kappa)$
(ii) $MI(\kappa)$

Note that the annoying mauves in text is a

(i) $\Rightarrow$ (ii). Take ultrapower $V_\lambda{}^\kappa/U$.
Show it's wellfounded.
Take Mostowski collapse M.
Then $j(x) := (\mathrm{const}_x)$ is elementary with $\mathrm{crit}(j)=\kappa$.

(ii) $\Rightarrow$ (i). Define
$$U := \{A\subseteq\kappa\,;\ \kappa\in j(A)\}$$
and show that it is a κ-complete nonprincipal uf. on κ.

We'll show: U as in (ii) $\Rightarrow$ (i) is always normal.

<u>Theorem</u> The ultrafilter

$$U = \{A \subseteq \kappa ;\ \kappa \in j(A)\}$$

is a normal ultrafilter.

<u>Proof.</u> As opposed to the proof of κ-completeness, now $\vec{A} = \{A_\alpha ; \alpha < \kappa\}$ is a sequence of length κ, so

REMINDER:
normal = closed under diagonal intersections

$$\underset{\alpha \leq \kappa}{\Delta} A_\alpha := \{\xi ;\ \xi \in \bigcap_{\alpha < \xi} A_\alpha\}$$

$j(\vec{A})$ is a seq. of length $j(\kappa)$

As before:

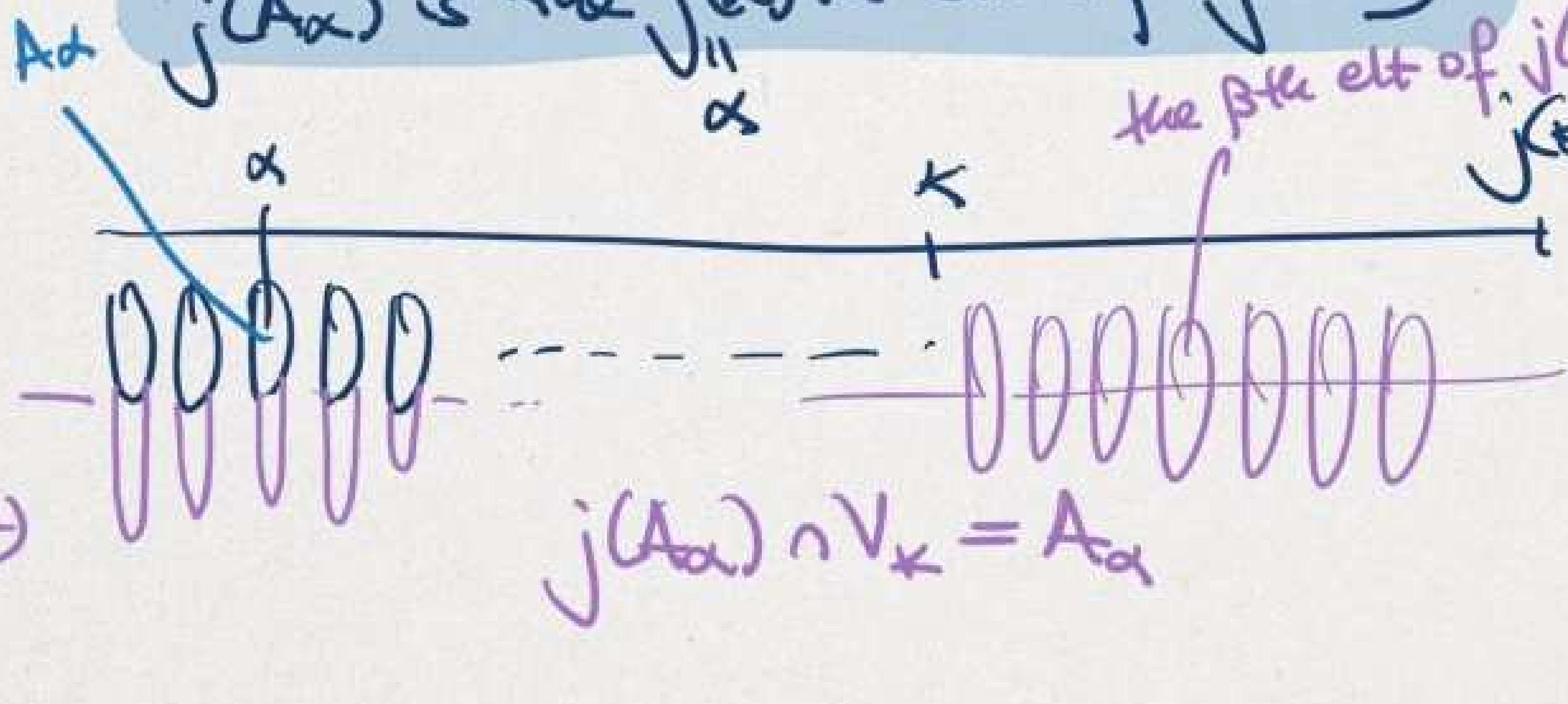
$\vec{A}$ is a seq. of subsets of κ of length κ

$j(\vec{A})$ is a seq. of subsets of $j(\kappa)$ of length $j(\kappa)$

*

A_α is the αth elt of $\vec{A}$

$j(A_\alpha)$ is the $\underset{\alpha}{\underset{\|}{j(\alpha)}}$th elt of $j(\vec{A})$



$j(A_\alpha) \cap V_\kappa = A_\alpha$

We need to show

$$\underset{\alpha \le \kappa}{\Delta} A_\alpha \in U$$

$$\iff \kappa \in j\left(\underset{\alpha \le \kappa}{\Delta} A_\alpha\right).$$

The formula that defines $\xi \in \underset{\alpha \le \kappa}{\Delta} A_\alpha$:

$$\xi \in \underset{\alpha \le \kappa}{\Delta} A_\alpha \iff \xi \in \bigcap_{\alpha < \xi} A_\alpha.$$

$\iff \forall \alpha < \xi$ ξ is in the αth elt of $\vec{A}$.

Therefore $\xi \in j\left(\underset{\alpha \le \kappa}{\Delta} A_\alpha\right) \iff$

$\forall \alpha < \xi$ ξ is in the αth elt of $j(\vec{A})$.

In particular, since $\kappa \in j(A_\alpha)$ for each $\alpha < \kappa$, we have $\kappa \in \bigcap_{\alpha < \kappa} j(A_\alpha)$.

Since the αth elt of $j(\vec{A})$ is $j(A_\alpha)$, this proves that

$$\kappa \in j\left(\underset{\alpha \le \kappa}{\Delta} A_\alpha\right). \qquad \text{q.e.d.}$$

Remark.

This is a way to normalize an ultrafilter:

Start with arbitrary U on κ, form j_U and take

$$U_{j_U}.$$

Then U_{j_U} is always normal.

Corollary Every measurable cardinal has a normal κ-complete uf. on it.

[ES#3 has an alternative construction of normal ufs.]

From Example Sheet #2

(29) Let F be a filter on a cardinal κ. Say that for $X \subseteq \kappa$, a function $f : X \to \kappa$ is called *regressive* if $f(\alpha) < \alpha$ for all $0 \neq \alpha \in X$. A set S is called F*-stationary* if for all $X \in F$, we have that $X \cap S \neq \varnothing$. Prove that the following statements are equivalent for a filter F.

(i) The filter F is closed under diagonal intersections.

(ii) For any F-stationary set S and any regressive $f : \cancel{X}\ S \to \kappa$, there is an $\alpha < \kappa$ such that $f^{-1}(\{\alpha\})$ is F-stationary.

TYPO

We get that:

Theorem TFAE

(i) U is normal

(ii) $[\mathrm{id}] = \kappa$.

Proof. (i) $\Rightarrow$ (ii).

Consider (29) in the context of ultrafilters.

Let U be an ultrafilter. Then

X U-stationary $\Longleftrightarrow$ $X \in U$.

[$\Leftarrow$ trivial; $\Rightarrow$ if X is U-stationary then $\kappa \setminus X \notin U$, so $X \in U$.]

Thus: U normal $\Longrightarrow$ $\forall X \in U\ \forall f : X \xrightarrow[\text{regressive}]{} \kappa$

$\exists \alpha$ s.t. $f^{-1}(\{\alpha\}) \in U$.

Let f be arbitrary s.t. $[f] \in [\mathrm{id}]$.

$\Longleftrightarrow \underbrace{\{\xi ; f(\xi) < \xi\}}_{S :=} \in U$.

So S is U-stationary & f is regressive on S.

Thus, there is $\alpha < \kappa$ s.t. $f^{-1}(\{\alpha\}) \in U$.

$$f^{-1}(\{\alpha\}) = \{\xi ; f(\xi)=\alpha\} \in U.$$

So $(f) = (\mathrm{const}_\alpha)$.

Thus for all $x \in (id)$, there is $\alpha < \kappa$ s.t.

$$x = \alpha.$$

Thus $(id) = \kappa$.

(ii) $\Rightarrow$ (i) will be on ~~Example Sheet~~ #3.

q.e.d.

NORMALITY GIVES US A STRENGTHENING OF REFLECTION

REFLECTION Properties φ of κ reflect below κ:

$$V_\lambda \models \varphi(\kappa)$$

& φ is absolute between V_λ & M

$$\Longrightarrow M \models \varphi(\kappa)$$

$$\Longrightarrow M \models \exists \mu < j(\kappa)\ \varphi(\mu)$$

$$V_\lambda \models \exists \mu < \kappa\ \ \varphi(\mu)$$

Boot-strapping allows to prove

$$V_\lambda \models \forall \alpha \exists \mu\ (\alpha < \mu < \kappa \wedge \varphi(\mu))$$

We used this to show, e.g., if κ is measurable, there are κ many inaccessibles below κ.

Strengthening with a normal U:

Prop. If U is normal, then

$$\{\mu < \kappa;\ \mu \text{ is inaccessible}\} \in U.$$

Proof. $M \models \kappa$ is inaccessible $\Longleftrightarrow M \models (id)$ is inacc.

$$\Longleftrightarrow \{\xi;\ id(\xi) \text{ is inacc.}\} \in U$$

$$\Longleftrightarrow \{\xi;\ \xi \text{ is inacc.}\} \in U.$$

q.e.d.

Application

If you have properties P_n for $n \in \mathbb{N}$ that κ has in M, they all reflect below κ.

Thus if U is normal,

$$X_n := \{ \xi ; \ \xi \text{ has } P_n \} \in U$$

and so $\bigcap_{n \in \mathbb{N}} X_n \in U$ by κ-completeness.

THAT PESKY INACCESSIBLE

Suppose κ is measurable, U is ultrafilter and we're trying to build the ULTRAPOWER OF UNIVERSE.

Problem 1 In this setting $[f]$ is a proper class, not a set.
Thus V^κ/U is a class consisting of classes.

Solution Scott's Trick.

$$[\![f]\!] := [f] \cap V_\alpha \text{ where } \alpha \text{ is least s.t. } V_\alpha \cap [f] = \emptyset.$$

Then define $V^\kappa/U := \{[\![f]\!] ; f : \kappa \longrightarrow V\}$

This is now a proper class whose elts are sets.

Problem 2 Elementarity requires a truth function for the universe.

In ZFC, for any given set M, the set $\{\varphi ; M \models \varphi\}$ is definable.

TARSKI'S UNDEFINABILITY OF TRUTH says that in general, no structure has a definable truth function for itself.

So, there is no T s.t.

$$V \models T(\varphi) \iff V \models \varphi.$$

SOLUTION By our solution to the 1st problem, we know that M and j are definable classes.

Now relativise formula to M:

$$\varphi \longmapsto \varphi^M \quad \text{where}$$

$$(\exists x \varphi) \longmapsto \exists x (x \in M \wedge \varphi^M)$$

then $M \models \varphi \iff \varphi^M$ holds.

$$E^U_\varphi :\iff \forall x_1 \dots \forall x_n \, \varphi(x_1, \dots, x_n) \longleftrightarrow \varphi^M(j(x_1), \dots j(x_n))$$

The previous proof gives:

$$\boxed{\begin{array}{c} \kappa \text{ is measurable } \& \ U \text{ uf.} \\ \Longrightarrow \\ E^U_\varphi \text{ holds for every } \varphi \end{array}}$$

The FUNDAMENTAL THEOREM ON MEASURABLE CARDINAL cannot even be expressed in the language of ZFC:

It would be something like:

κ is measurable $\iff \underbrace{\exists M \exists j}_{\text{not set quantifiers.}} E^{\psi}_{\varphi}$ for all φ

↳ CLASS THEORIES

Two-sorted theories with two sorts of variables: <u>sets</u> and <u>classes</u>.
[or a definable predicate Set(x).]

von Neumann-Bernays-Gödel

NBG.

John von Neumann 1903–1957

Paul Bernays 1888–1977

Kurt Gödel 1906–1978

<u>Roughly</u>: In NBG, classes are just definable classes, i.e., if $M \models ZFC$ and $C := \{X \subseteq M;\ X \text{ is definable}\}$, then $(M, C) \models NBG$

Competing theory: <u>Kelley-Morse</u>

KM

In KM, classes are fundamentally different from sets, e.g., $(M, \mathcal{P}(M))$ would be a model of KM