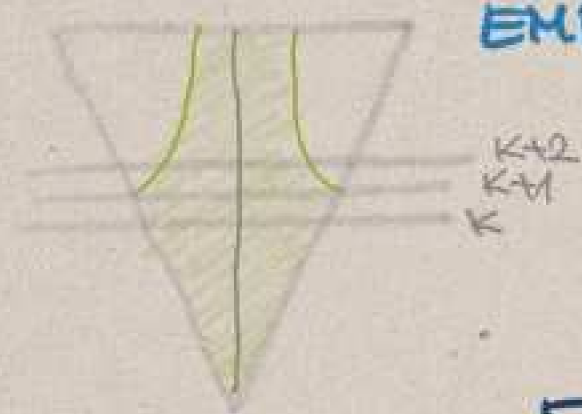
XII TWELFTH LECTURE OF LARGE CARDINALS

1 March 2023

DEFINITION

κ is an EMBEDDING CARDINAL

$E(\kappa) :\Longleftrightarrow \exists\lambda \exists M \exists j$ $\kappa < \lambda$ & λ is inaccessible & $M \subseteq V_\lambda$ & $j : V_\lambda \longrightarrow M$ & M is transitive & j is elementary & $\kappa = \mathrm{crit}(j)$



$E(\kappa, M) :\Longleftrightarrow$ M witnesses $E(\kappa)$

$MI(\kappa) :\Longleftrightarrow \exists\lambda\ \kappa < \lambda$ & $I(\lambda)$ & $M(\kappa)$

Theorem (Lectures X & XI)

$$MI(\kappa) \Longrightarrow E(\kappa)$$

More specifically: $E(\kappa, M)$ for any M that is the Mostowski collapse of V_λ^κ / U for a κ-complete nonprincipal uf. on κ.

Today:

IS κ MEASURABLE IN M? (*)

INTUITIVE META-ARGUMENT:

If we start with a model with but one measurable and $M \models j(\kappa)$ measurable and $M \models \kappa$ measurable, then we created a measurable out of thin air!
That should not be possible!

Remark: Just the fact that $U \notin M$ does not show that $M \models \kappa$ is not measurable. There could be other uf. U' on κ s.t. $U' \in M$.
So, combinatorially, the question (*) means: Are those two FUNDAMENTALLY DIFFERENT uf. on κ?

Making the meta-argument precise.

APPROACH 1

$$\mathrm{MIC} := \exists\kappa\, \mathrm{MI}(\kappa)$$
$$\iff \exists\kappa\exists\lambda\ \kappa<\lambda\ \&\ M(\kappa)\ \&\ I(\lambda)$$

2MC
$$\iff \exists\kappa\exists\lambda\ \kappa<\lambda\ \&\ M(\kappa)\ \&\ M(\lambda)$$

We have $\mathrm{MIC} <_{\mathrm{Cons}} \mathrm{2MC}$

[Proof. Suppose 2MC, have unboundedly many inacc. between κ & λ. Let λ' be the 2nd one.
Then $V_{\lambda'} \models \mathrm{MIC}$, so Cons(MIC).]

Theorem If MIC is consistent, then

$\oplus$ $E(\kappa, M) \not\Longrightarrow M \models \kappa$ is measurable.

Proof. Work in MIC; let κ be least measurable & λ be least inaccess. above κ.
By Lectures $\overline{\mathrm{X}}$ & $\overline{\mathrm{XI}}$ form transitive M from uf. U on κ in V_λ. In particular, $E(\kappa, M)$ holds.

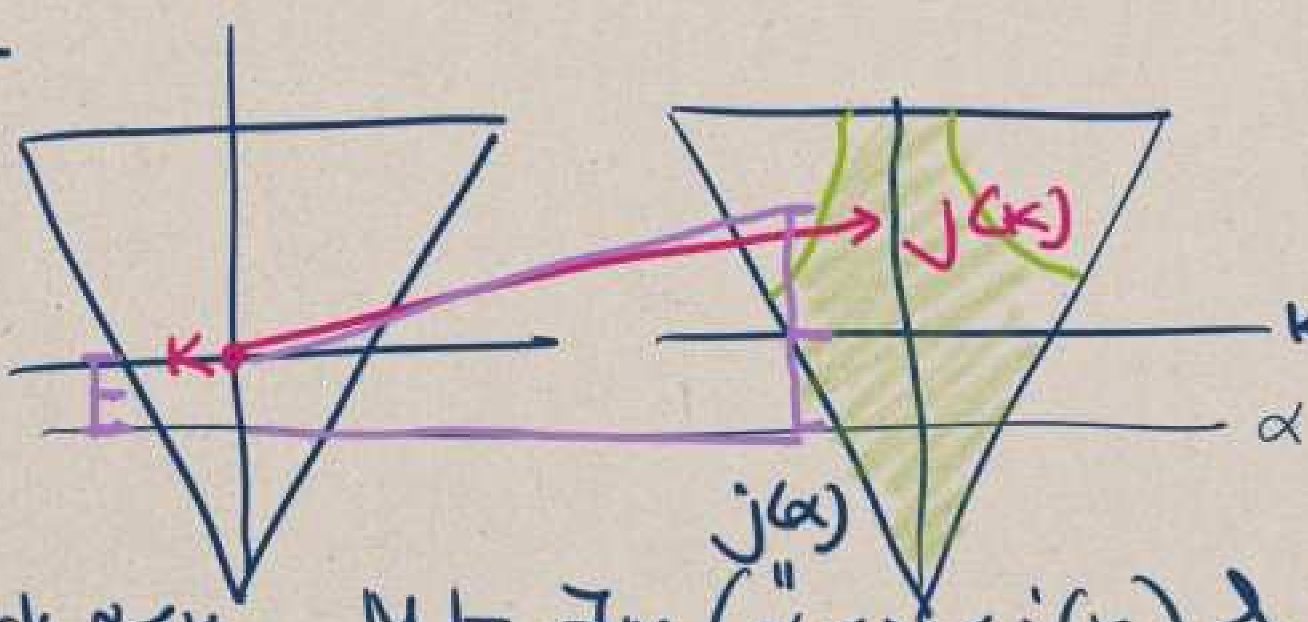
So, assume towards contr. that $\neg\oplus$ is true.
Thus $M \models \kappa$ is meas. & $j(\kappa) > \kappa$ is meas.
So $M \models \mathrm{2MC}$. $\Longrightarrow$ we proved Cons(MIC) in contradiction to G2. q.e.d.

APPROACH 2

Def. κ is surviving if there is $\lambda > \kappa$
$I(\lambda)$, $M \subseteq V_\lambda$ transitive,
$j: V_\lambda \longrightarrow M$,
$\kappa = \mathrm{crit}(j)$ and
$M \models \kappa$ is measurable.

Theorem If κ is surviving, then there are unboundedly many measurable cardinals below κ.
In particular, the smallest measurable (2nd smallest, 3rd smallest etc) cannot be surviving.

Proof.



For each $\alpha<\kappa$ $\quad M \models \exists\mu\,(\overset{j(\alpha)}{\overset{\shortparallel}{\alpha}} < \mu < j(\kappa) \ \&\ M(\mu))$

$\overset{\text{elem.}}{\Longrightarrow} \quad V_\lambda \models \exists\mu\,(\alpha < \mu < \kappa \ \&\ M(\mu))$

FUNDAMENTAL THEOREM ON MEASURABLE CARDINALS

Theorem Let κ be an uncountable cardinal.

TFAE: (i) $E(\kappa)$
(ii) $MI(\kappa)$

Note that the annoying macros in this is a technical nuisance: cf. Lecture $\underline{\text{XIII}}$.

Proof. (ii) $\Rightarrow$ (i) was done in lectures $\overline{\underline{\text{X}}}$ & $\overline{\underline{\text{XI}}}$.

(i) $\Rightarrow$ (ii). Suppose $E(\kappa)$,

so $j : V_\lambda \longrightarrow M$ $\quad j \restriction \kappa = id$, $\quad j(\kappa) > \kappa$

Define $U := U_j := \{ A \subseteq \kappa ; \kappa \in j(A) \}$

Note: $A \subseteq \kappa \Rightarrow j(A) \subseteq j(\kappa)$.

Claim U is a κ-complete u.f. on κ.

(1) $\kappa \in U$ since $\kappa \in j(\kappa)$.

(2) $\emptyset \notin U$ since $\kappa \notin j(\emptyset) = \emptyset$.

(3) $A \subseteq B \Rightarrow j(A) \subseteq j(B)$ by elt.

So if $A \in U \Rightarrow \kappa \in j(A) \subseteq j(B)$

$\Rightarrow B \in U$.

④ Since $j(\kappa \setminus A) = j(\kappa) \setminus j(A)$,
we have that
$$\kappa \notin j(A) \Longrightarrow \kappa \in j(\kappa \setminus A),$$
so U is ultra.

⑤ U is nonprincipal since
$$j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\}.$$
Thus $\kappa \notin j(\{\alpha\})$, so U is nonprincipal.

⑥ κ-completeness
Let $\gamma < \kappa$ and $\vec{A} = \{A_\alpha ; \alpha < \gamma\} \subseteq U$.
[i.e., $\kappa \in j(A_\alpha)$ f.a. $\alpha < \gamma$]

To show κ-completeness, need to show:
$$\kappa \in j\Big(\bigcap_{\alpha<\gamma} A_\alpha\Big) \boxed{= \bigcap_{\alpha<\gamma} j(A_\alpha)}$$

$\beta \in \bigcap_{\alpha<\gamma} A_\alpha \iff \forall \alpha < \gamma$ β is in the αth elt of $\vec{A}$

$\beta \in j(\bigcap_{\alpha<\gamma} A_\alpha) \iff \forall \alpha < \underset{\gamma}{\underset{\shortparallel}{j(\gamma)}}$ β is in the αth elt. of $j(\vec{A})$

A_α is the αth elt of $\vec{A}$
$\Longrightarrow j(A_\alpha)$ is the $\underset{\alpha}{\underset{\shortparallel}{j(\alpha)}}$th elt of $j(\vec{A})$

Thus $\kappa \in \bigcap_{\alpha<\gamma} j(A_\alpha)$ and that was to show.
q.e.d.

Corollary to the proof of the FTMC

If U_j is constructed as in the proof, then U_j is normal.

[Closed under diagonal intersections:

$$\underset{\alpha\leq\kappa}{\triangle} A_\alpha := \{\gamma ; \gamma \in \bigcap_{\alpha<\gamma} A_\alpha\}]$$

Remark This indicates that the operations

$$U \longmapsto j_U \ \& \ j \longmapsto U_j$$

from the proof of FTMC are not inverses. In particular U_{j_U} is always normal independent of whether U was.

Proof. Next time.

Problem is that $\vec{A} = \{A_\alpha ; \alpha<\kappa\}$ is a sequence of length κ, so by elementarity

$j(\vec{A})$ is a sequence of length $j(\kappa) > \kappa$.

As a consequence, we cannot easily write down $j(\vec{A})$ in terms of the original A_α as we did in the proof of κ-completeness.

This will be discussed in Lecture $\underline{XIII}$.