XI Large Cardinals

ELEVENTH LECTURE

27 February 2023

IMPORTANT NOTICE

There was a missing step in the argument of the proof of page 4 in Lecture X.

Main difference: One of $X^0, X^1 \in U$, w.l.o.g. X^0

$$H := X^0 \cap \underset{\alpha<\kappa}{\triangle} X_\alpha .$$

The proof in the hand-written lecture notes on moodle has been corrected:

NOTE: THERE WAS A TECHNICAL ERROR ON P. 4 IN THE LECTURES: CORRECTED

Let $X^0 := \{\alpha ; b(\alpha) = 0\}$
& $X^1 := \{\alpha ; b(\alpha) = 1\}$
One of these is in U.
Assume $X^0 \in U$ and show that then there is a homogeneous set for colour 0.
[The case $X^1 \in U \Longrightarrow$ there is a homogeneous set for colour 1 is identical.]

Let
$$X_\alpha := \begin{cases} X^0_\alpha & \text{if } b(\alpha) = 0 \\ \kappa & \text{o/w} \end{cases}$$

Clearly, $X_\alpha \in U$ for all α. Thus
$$\underset{\alpha<\kappa}{\triangle} X_\alpha \in U \quad \text{by normality,}$$
and thus
$$H := X^0 \cap \underset{\alpha<\kappa}{\triangle} X_\alpha \in U.$$

Since U was κ-complete & nonprincipal, $|H| = \kappa$, so if we can show that H is homogeneous for f, we're done.

<u>Claim</u>: H is homogeneous for colour 0.
Let $\gamma < \delta$ with $\gamma, \delta \in H$. Since $\gamma, \delta \in H$, we have $X_\gamma = X^0_\gamma$, $X_\delta = X^0_\delta$.

Then $\delta \in H \Longrightarrow \delta \in \underset{\alpha<\kappa}{\triangle} X_\alpha \Longleftrightarrow \delta \in \bigcap_{\alpha<\delta} X_\alpha \subseteq X_\gamma = X^0_\gamma$.

So $\delta \in X^0_\gamma \Longrightarrow f_\gamma(\delta) = 0 \Longrightarrow f(\{\gamma, \delta\}) = 0$.
q.e.d.

ULTRAPOWERS OF THE UNIVERSE

The Universe: V_λ [where λ is inaccessible]

Assume $V_\lambda \models ZFC + MC$.

Let κ be the measurable.

$N := (V_\lambda)^\kappa / U$; (1) $N \models ZFC + MC$.

(5) $x \longmapsto [\text{const}_x]$

is elementary embedding from V_λ to N

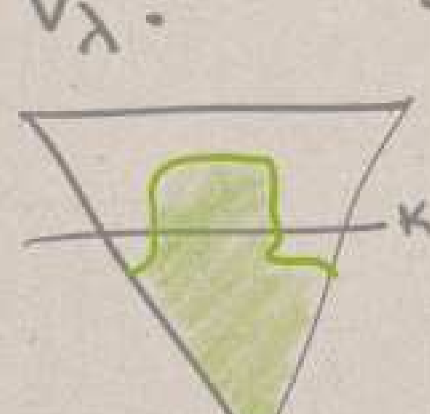
(6) N is wellfounded

(7) There is a transitive set $M \cong N$.

? No: M cannot exceed V_λ!

$j : V_\lambda \longrightarrow M$ elementary embedding

(8) $M \subseteq V_\lambda$.



Can M look like this?

TODAY, we'll see that the answer is negative.

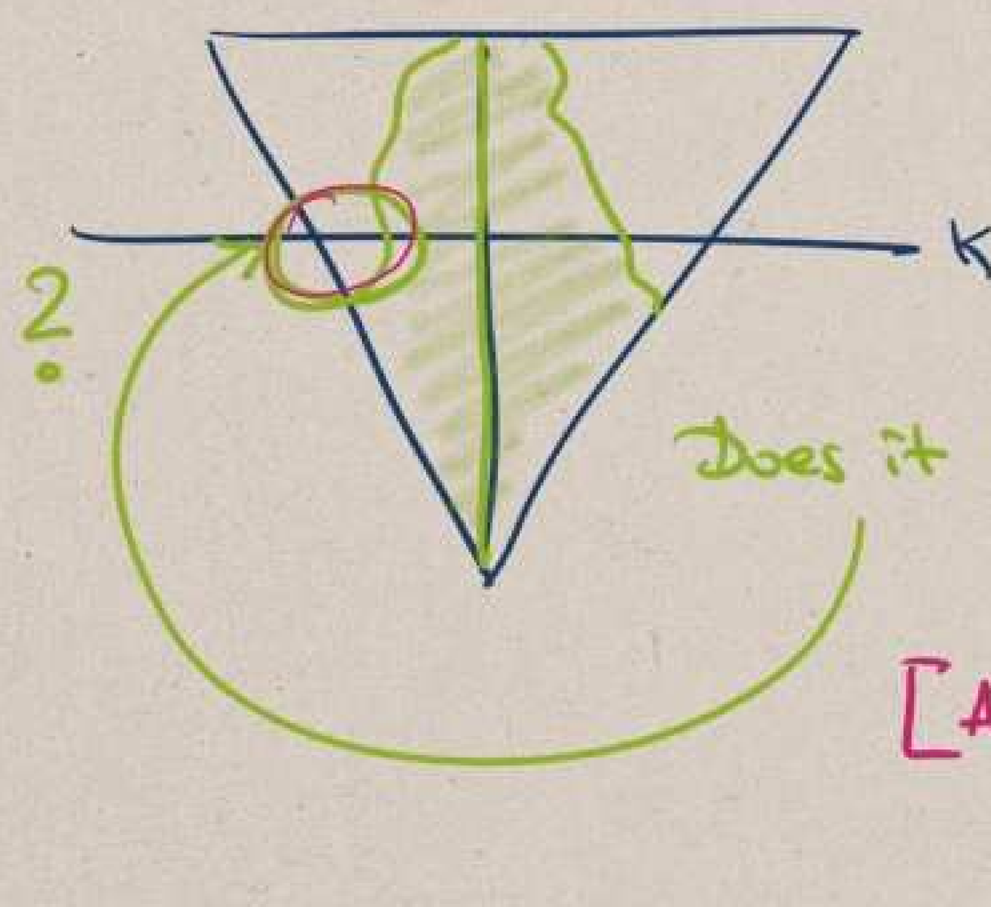
(19) Claim: $\mathrm{Ord} \cap M = \lambda$.

Since j is elementary

$$M \models j(\alpha) \text{ is an ordinal}$$
for each $\alpha < \lambda$.

Since j is order preserving,
$$\mathrm{ot}(\{j(\alpha) ; \alpha < \lambda\}) = \lambda.$$

So $\mathrm{Ord} \cap M$ is a transitive subset of λ of o.t. λ, thus $\mathrm{Ord} \cap M = \lambda$.



Does it look like this ?

[Answer: No!]

(10) Claim $j \restriction V_\kappa = \mathrm{id}$.

We can show this by $\in$-induction.

Suppose $x \in V_\kappa$ s.t. for all $y \in x$, $j(y) = y$.

Let $z \in x \in V_\kappa$.

$$\overset{\text{Elem.}}{\Longrightarrow} \underset{\overset{\| \text{ IH}}{z}}{j(z)} \in j(x).$$

So $\boxed{x \subseteq j(x).}$ (*)

Suppose now that $z \in j(x)$.

Let f be s.t. $z = (f)$. Then

$$(f) \in (\mathrm{const}_x)$$

By Łoś: $\{\alpha ; f(\alpha) \in \underbrace{\mathrm{const}_x(\alpha)}_{= x}\} \in U.$

$$= \bigcup_{y \in x} \{\alpha ; f(\alpha) = y\}.$$

size $|x| < \kappa$ union, since κ is measurable.

By κ-completeness, there is $y \in x$ s.t.

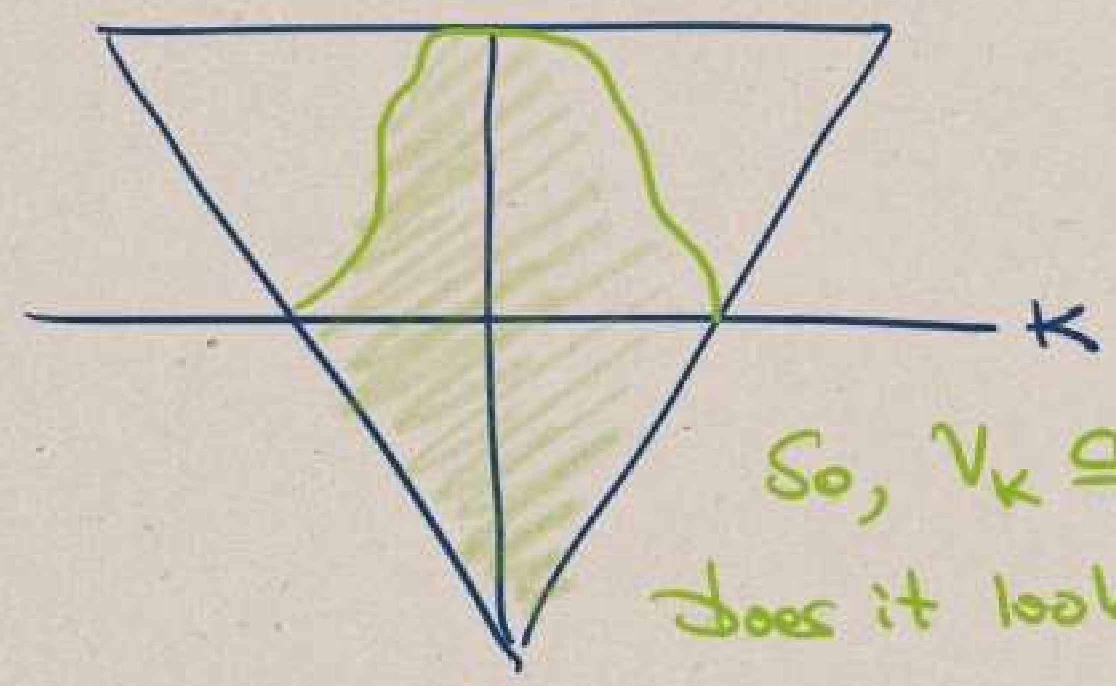
$$\{\alpha ; f(\alpha) = y\} \in U.$$

So $f \sim_U \mathrm{const}_y$.

$$(f) = j(y) \overset{\text{IH}}{=} y.$$

So $j(x) \subseteq x$. With (*), get

$x = j(x)$.



So, $V_\kappa \subseteq M$.

Does it look like that?

(11) $j \neq \mathrm{id}$ More concretely

$$j(\kappa) > \kappa.$$

$j(\kappa) = (\mathrm{const}_\kappa)$; by (10) if $\alpha < \kappa$, $j(\alpha) = (\mathrm{const}_\alpha)$.

Consider $\mathrm{id} : \kappa \longrightarrow \kappa$

$$\{\alpha ; \mathrm{id}(\alpha) < \mathrm{const}_\kappa(\alpha)\}$$
$$= \{\alpha ; \alpha < \kappa\} = \kappa \in U$$

So, by Łoś,

$$(\mathrm{id}) < (\mathrm{const}_\kappa) = j(\kappa).$$

But, fix $\gamma < \kappa$

$$\{\alpha;\ \mathrm{id}(\alpha) < \mathrm{const}_\gamma(\alpha)\}$$
$$= \{\alpha;\ \alpha < \gamma\} \notin U$$ since U is κ-complete & non-principal

So $\{\alpha; \alpha \geq \gamma\} \in U$

So, by Łoś, $(\mathrm{id}) \geq (\mathrm{const}_\gamma) \underset{10}{=} \gamma$.

Thus $\kappa \leq (\mathrm{id}) < j(\kappa)$.

[Later: consider under what extra assumptions, we get $(\mathrm{id}) = \kappa$.]

Def. We say κ is the critical point of j if $j \restriction \kappa = \mathrm{id}$ & $j(\kappa) \neq \kappa$.

So, we just proved: κ is the critical point of j_U.

(12) Claim: $V_{\kappa+1} \subseteq M$

[Note: it's not possible by (11) that $j \upharpoonright V_{\kappa+1} = id$.]

If $A \in V_{\kappa+1}$, i.e., $A \subseteq V_\kappa$, then

$$A = j(A) \cap V_\kappa \in M.$$

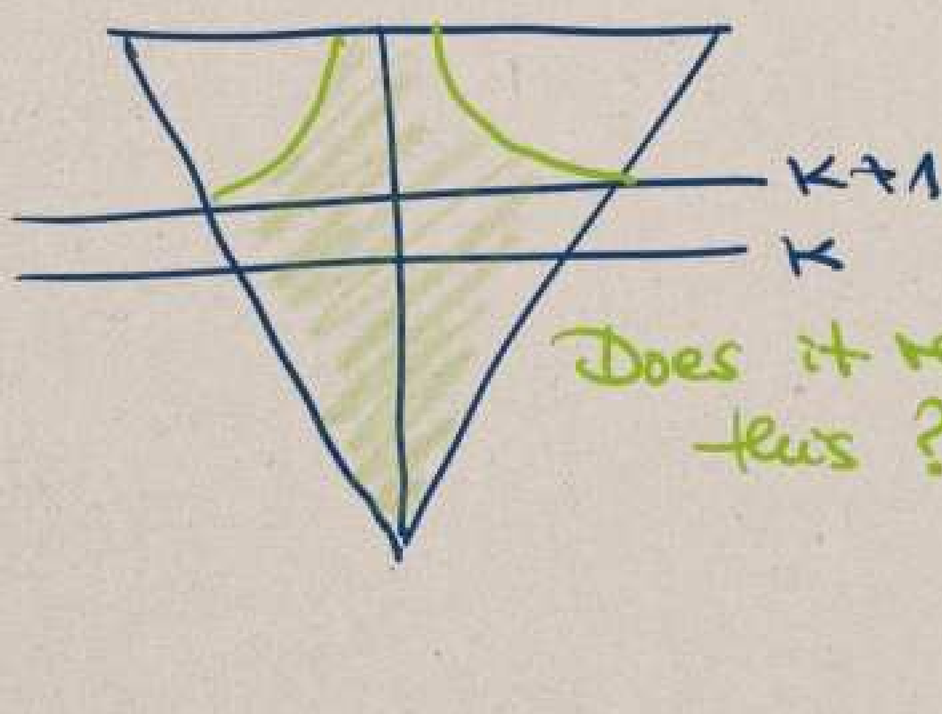
[Proof. $\underline{x \in A \subseteq V_\kappa} \overset{(10)}{\Longrightarrow} j(x) = x$

$\overset{\text{elem.}}{\Longrightarrow} x = j(x) \in j(A)$

$\Longrightarrow x \in j(A) \cap V_\kappa$

$x \in j(A) \cap V_\kappa \overset{(10)}{\Longrightarrow} x = j(x)$ and thus $j(x) \in j(A)$

$\overset{\text{elem.}}{\Longrightarrow} x \in A.$]



Does it really look like this?

(13) General absoluteness tells us that w.l.o.g., we could assume that λ is the least inaccessible above κ.

GENERAL ABSOLUTENESS OBSERVATION

Let $A \subseteq V_\lambda$ transitive.

(a) If $V_{\mu+1} \subseteq A$, then "μ is inaccessible" is absolute between A and V_λ.

(b) If $V_{\mu+2} \subseteq A$, then "μ is measurable" is absolute between A and V_λ.

If not, let λ^* be least inaccessible above κ and just work in V_{λ^*}.

But then $V_{\lambda^*} \models j(\kappa)$ is not measurable.

But by elem. $M \models j(\kappa)$ is measurable.

Thus $M \neq V_{\lambda^*}$.

(14) Let's give an argument that does not depend on choice of λ / λ^*.

Remember (2) & (8):

$$V_\lambda \models |\{\alpha(f)\,;\, (f) \in \mu\}| \leq |V_{\mu+1}| < \lambda.$$
$$\mu \geq \kappa$$

If $\mu = j(\kappa) = (\mathrm{const}_\kappa)$, then

$$V_\lambda \models |\{(f);\ (f) < j(\kappa)\}| \leq$$

$$|\{f;\ f : \kappa \longrightarrow \kappa\}| = 2^\kappa.$$

$\Longrightarrow$ $V_\lambda \models j(\kappa)$ is not a strong limit cardinal

$\Longrightarrow$ $V_\lambda \models j(\kappa)$ is not inaccessible

$\Longrightarrow$ $V_\lambda \models j(\kappa)$ is not measurable.

And again: $M \neq V_\lambda$.

cf. page 8

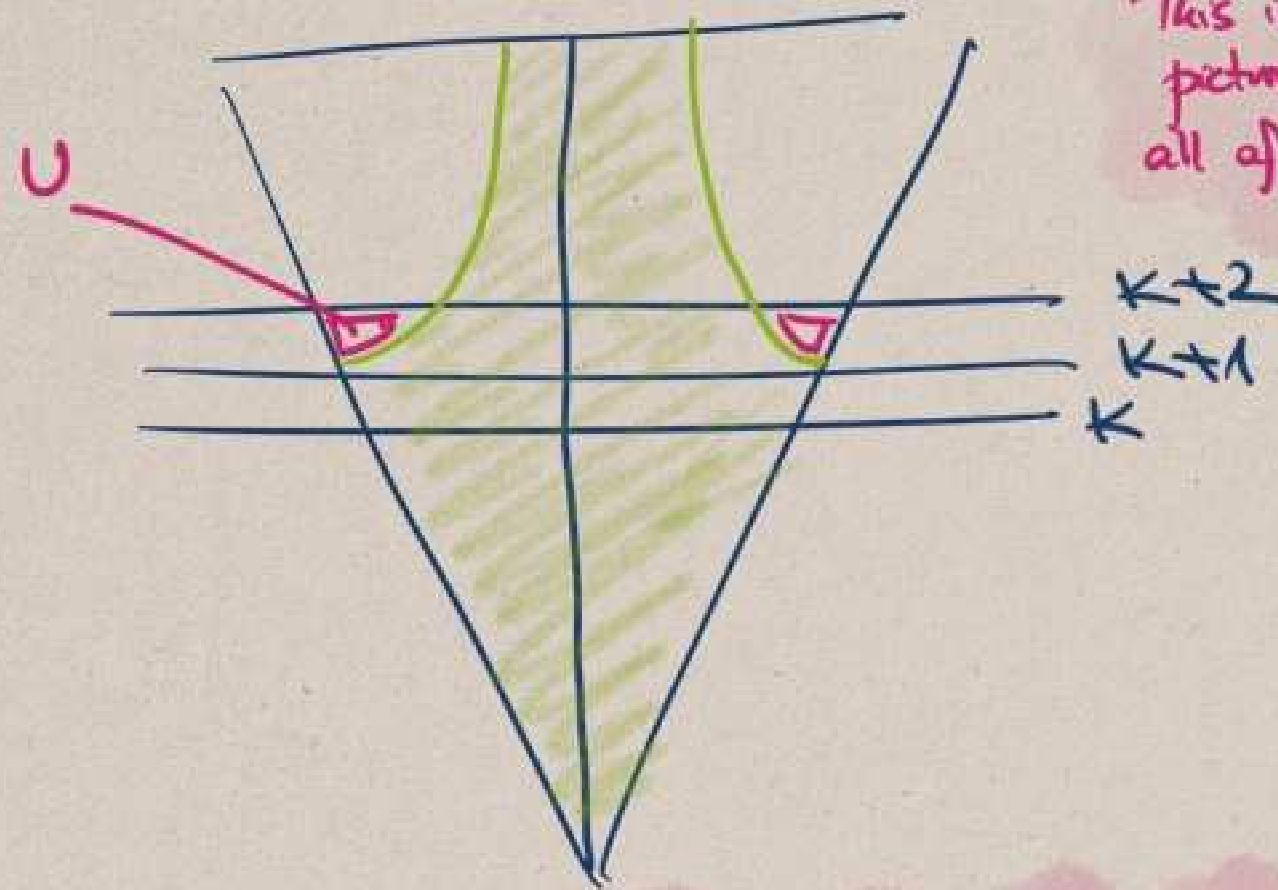
(15) By general absoluteness, this implies $V_{j(\kappa)+2} \not\subseteq M$.

(16) We can do better and show that

$U \notin M$. $\boxed{\text{ES\#3}}$

So $V_{\kappa+2} \not\subseteq M$.

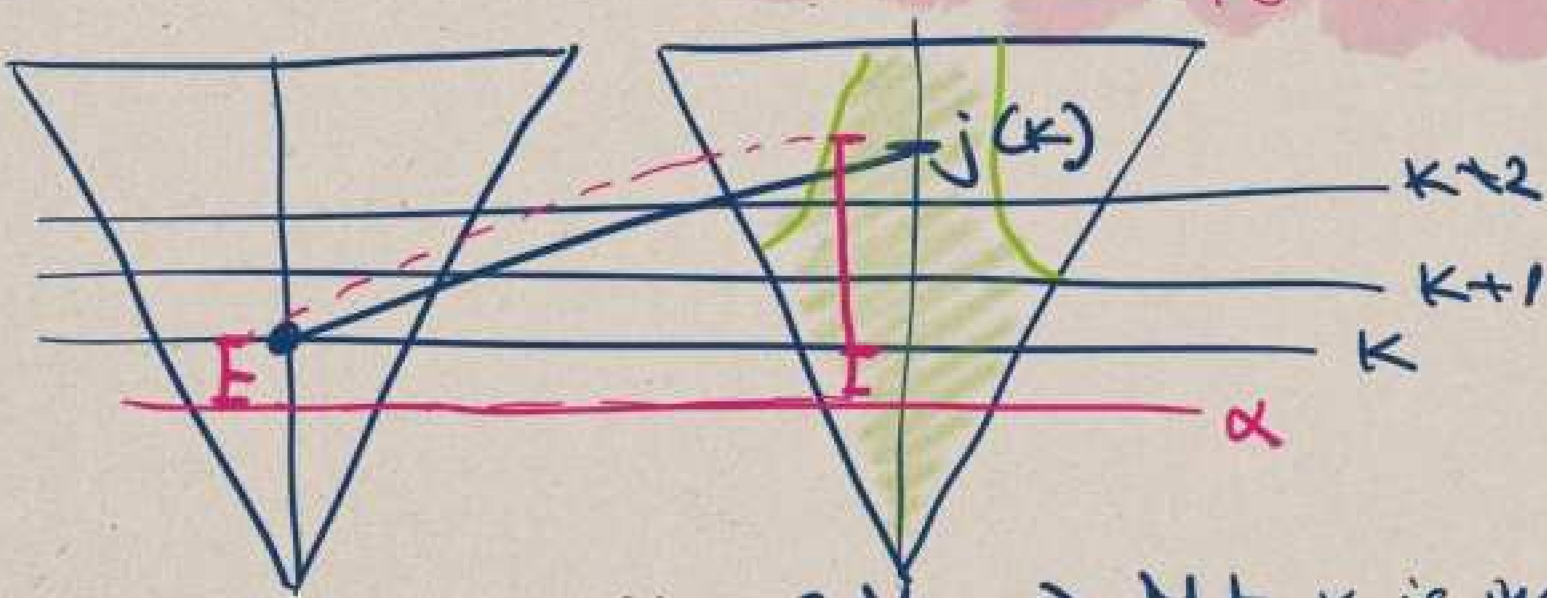
Essentially repeating (14) in any model containing U.



This is the correct picture: M includes all of κ+1 and misses something in $V_{\kappa+2}$.

REFLECTION

Remember our discussion of reflection using the Keisler extension property (KEP): Lecture IX, pages 5 & 6.



$V_{\kappa+1} \subseteq M \Longrightarrow M \models \kappa$ is inaccessible

$$M \models \exists \mu \, (\underset{j(\alpha)}{\underset{\parallel}{\alpha}} < \mu < j(\kappa) \;\&\; \mu \text{ is inaccessible})$$

$$V_\kappa \models \exists \mu \, (\alpha < \mu < \kappa \;\&\; \mu \text{ is inaccessible})$$

This is an alternative proof of

κ measurable

$\Longrightarrow$

there are unboundedly many inaccessibles below κ

[Previous proof: Lecture VIII

κ meas. $\Longrightarrow$ κ weakly compact

κ w.c. $\Longrightarrow$ unboundedly many inaccessibles.] Lecture IX