X

Large Cardinals
Tenth Lecture

22 February 2023

κ is a <u>finite</u> <u>partition</u> <u>cardinal</u>

if $\kappa \longrightarrow (\kappa)^2_2$.

κ measurable

$\Downarrow \longleftarrow$ today ✓

κ finite partition

$\Downarrow \longleftarrow$ Lecture <u>IX</u>

κ inaccessible

<u>Note</u> (without proof)

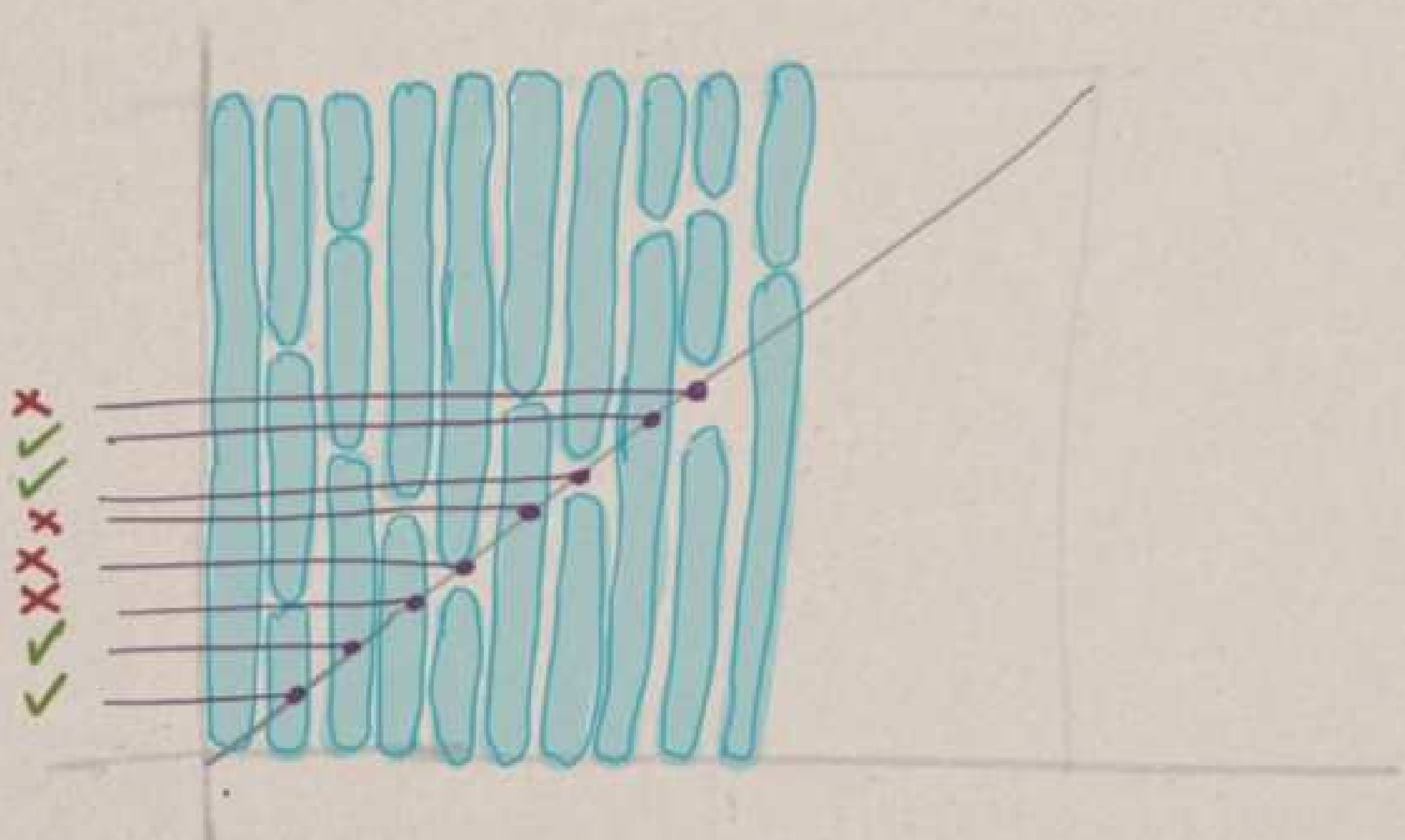
κ finite partition $\Longleftrightarrow$ κ weakly compact.

DEFINITION Let $\{A_\alpha ; \alpha < \kappa\}$ be a family of subsets of κ.

Then

$$\mathop{\triangle}_{\alpha<\kappa} A_\alpha := \left\{ \xi \; ; \; \xi \in \bigcap_{\alpha<\xi} A_\alpha \right\}$$

is called the **DIAGONAL INTERSECTION** of the family.



Def. A filter F on κ is called **NORMAL** if it is closed under diagonal intersections.

Theorem If κ is measurable and U is a normal, κ-complete nonprincipal uf. on κ, then κ is finite partition.

Background We'll show later ($^{\text{probably in Lecture}}_{\overline{\text{XI}} \text{ or } \overline{\text{XII}}}$) that every meas. cardinal carries such a filter.

Thus: Corollary Every meas. card. is finite partition.

Proof. Let $f : [\kappa]^2 \longrightarrow \{0,1\}$. We need to show that there is a homogeneous set of size κ.

Fix $\alpha < \kappa$, define

$$f_\alpha(\beta) := f(\{\alpha,\beta\}) \quad \left[\begin{array}{c}\text{undefined if} \\ \alpha = \beta\end{array}\right]$$

If α is fixed,

$$X_\alpha^0 := \{\beta ; f_\alpha(\beta) = 0\}$$

$$X_\alpha^1 := \{\beta ; f_\alpha(\beta) = 1\}$$

Then exactly one of these is in U.

Let $b(\alpha)$ be such that $X_\alpha^{b(\alpha)} \in U$.

Let $X^0 := \{\alpha; b(\alpha) = 0\}$
& $X^1 := \{\alpha; b(\alpha) = 1\}$
One of these is in U.
Assume $X^0 \in U$ and show that there there is a homogeneous set for colour 0.

NOTE: THERE WAS A TECHNICAL ERROR ON P. 4 IN THE LECTURES: CORRECTED

[The case $X^1 \in U \implies$ there is a homogeneous set for colour 1 is identical.]

Let
$$X_\alpha := \begin{cases} X^0_\alpha & \text{if } b(\alpha) = 0 \\ \kappa & \text{o/w} \end{cases}$$

Clearly, $X_\alpha \in U$ for all α. Thus
$$\underset{\alpha \leq \kappa}{\Delta} X_\alpha \in U \quad \text{by normality,}$$
and thus
$$H := X^0 \cap \underset{\alpha \leq \kappa}{\Delta} X_\alpha \in U.$$

Since U was κ-complete & nonprincipal, $|H| = \kappa$, so if we can show that H is homogeneous for f, we're done.

<u>Claim</u>: H is homogeneous for colour 0.
Let $\gamma < \delta$ with $\gamma, \delta \in H$. Since $\gamma, \delta \in H$, we have
$$X_\gamma = X^0_\gamma,\ X_\delta = X^0_\delta.$$

Then $\delta \in H \implies \delta \in \underset{\alpha \leq \kappa}{\Delta} X_\alpha \iff \delta \in \bigcap_{\alpha < \delta} X_\alpha \subseteq X_\gamma = X^0_\gamma$.

So $\delta \in X^0_\gamma \implies f_\gamma(\delta) = 0 \longrightarrow f(\{\gamma, \delta\}) = 0$.

q.e.d.

Measurable cardinals & elementary embeddings

[or: ULTRAPOWERS OF THE UNIVERSE]

Assume $\kappa < \lambda$ where κ is measurable
λ is inaccessible.

<u>Discussion</u> (a) This assumption is strictly stronger in consistency strength than MC.

[If the assumption holds, then
$V_\lambda \models ZFC + MC$,
so Cons(ZFC+MC) follows.

We call V_λ the UNIVERSE.

Why? If $V_{\kappa+2} \subseteq N$ &
$V_{\kappa+1}^{\alpha} \subseteq N$ for all $\alpha < \kappa$
then "is measurable" is absolute between V and N.
Clearly V_λ satisfies this.]

(b) We do this mostly for reasons of convenience. [Avoid talking about proper classes!]
Later: we'll sketch how to avoid the additional strength.

Let U be a κ-complete nonprincipal uf. on κ, let

$$M_\alpha := V_\lambda \quad \text{for all } \alpha.$$

and $N := \prod_{\alpha<\kappa} M_\alpha / U = (V_\lambda)^\kappa / U$.

[Called an ultrapower: ultraproduct with all factors the same.]

MANY OBSERVATIONS

(1) By Łoś, $N \models ZFC + MC$ because $\{\alpha ; V_\lambda \models ZFC+MC\} = \kappa \in U$.

(2) If $f : \kappa \longrightarrow V_\lambda$, then by regularity of λ, there is $\alpha < \lambda$ s.t. $f \in V_\alpha$.

Thus $[f] \subseteq V_\lambda$, however $[f] \notin V_\lambda$.

Each $[f]$ contains elements of arbitrary rank below λ.

(3) $N \models [f]$ is an ordinal

$$\overset{\text{Łoś}}{\Longleftrightarrow}$$

$\{\alpha ; f(\alpha) \text{ is an ordinal}\} \in U$.

Define $f'(\alpha) := \begin{cases} f(\alpha) & \alpha \text{ is ordinal} \\ 0 & \text{o/w} \end{cases}$ $\Rightarrow f' \sim_U f \Rightarrow [f] = [f']$

So w.l.o.g., if

$N \models [f]$ is an ordinal,

we may assume $f : \kappa \longrightarrow \lambda$.

(4) Suppose $f : \kappa \longrightarrow V_\lambda$; by (2), there is α s.t. $f \in V_\alpha$.

Consider g s.t. $[g] E [f]$

$\iff \{\gamma ; g(\gamma) \in f(\gamma)\} \in U$.

Define $g'(\gamma) := \begin{cases} g(\gamma) & \text{if } g(\gamma) \in f(\gamma) \\ 0 & \text{o/w} \end{cases}$.

So $g' \sim_U g$, so $[g'] = [g]$, but $g' \in V_\alpha$.

Thus $|\{[g] ; [g] E [f]\}| \leq |V_\alpha| < \lambda$.

[since λ was inaccessible]

(5) If $x \in V_\lambda$, consider $\mathrm{const}_x : \alpha \longmapsto x$

constant function.

Consider $x \longmapsto [\mathrm{const}_x]$.

$V_\lambda \longrightarrow N$

By Łoś, this is an elementary embedding.

$$V_\lambda \models \varphi(x_1, \dots, x_n)$$

$$\Longleftrightarrow V_\lambda \models \varphi(\mathrm{const}_{x_1}(\alpha), \dots, \mathrm{const}_{x_n}(\alpha))$$

for all α.

$$\Longleftrightarrow \{\alpha ; V_\lambda \models \varphi(\mathrm{const}_{x_1}(\alpha), \dots, \mathrm{const}_{x_n}(\alpha))\} = \kappa \in U$$

$$\overset{\text{Łoś}}{\Longleftrightarrow} N \models \varphi([\mathrm{const}_{x_1}], \dots, [\mathrm{const}_{x_n}]).$$

(6) N is wellfounded

Suppose not. Then there is seq. f_n $(n \in \mathbb{N})$ s.t. $N \models [f_{n+1}] \in [f_n]$ for all n.

$$\underset{\text{Łoś}}{\Longleftrightarrow} \underbrace{\{\alpha ; f_{n+1}(\alpha) \in f_n(\alpha)\}}_{= X_n} \in U.$$

Let $X := \bigcap_{n \in \mathbb{N}} X_n \in U$ [by κ-completeness]

So $X \neq \emptyset$, pick $\alpha \in X$; then $\{f_n(\alpha) ; n \in \mathbb{N}\}$ is a descending sequence in V.

(7) Thus by Mostowski, find transitive set M s.t.

$$(M,\in) \cong (N,E)$$

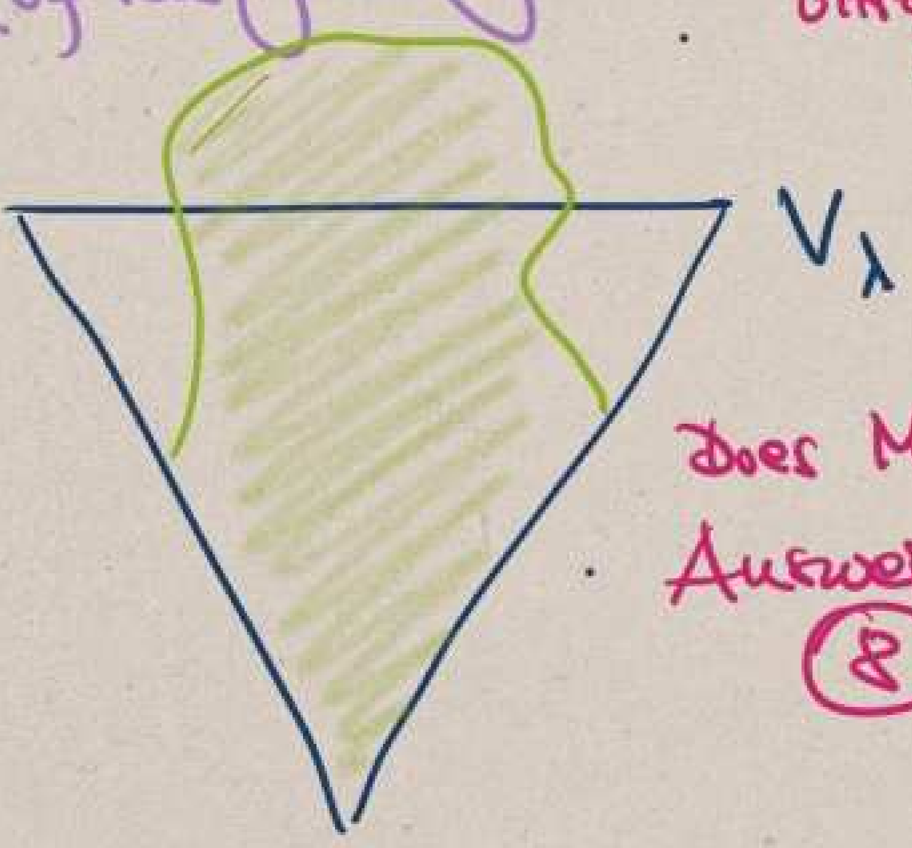
We write $\mathrm{Ult}(V_\lambda, U)$ for this structure $(M,\in)$, called the ultrapower of the universe.

Let (f) be the Mostowski pre-image of $[f]$

and $j_U(x) := (\mathrm{const}_x)$

called the ultrapower embedding

[often just $j(x)$ if there is only one ultrafilter U under consideration]



Does M look like this?

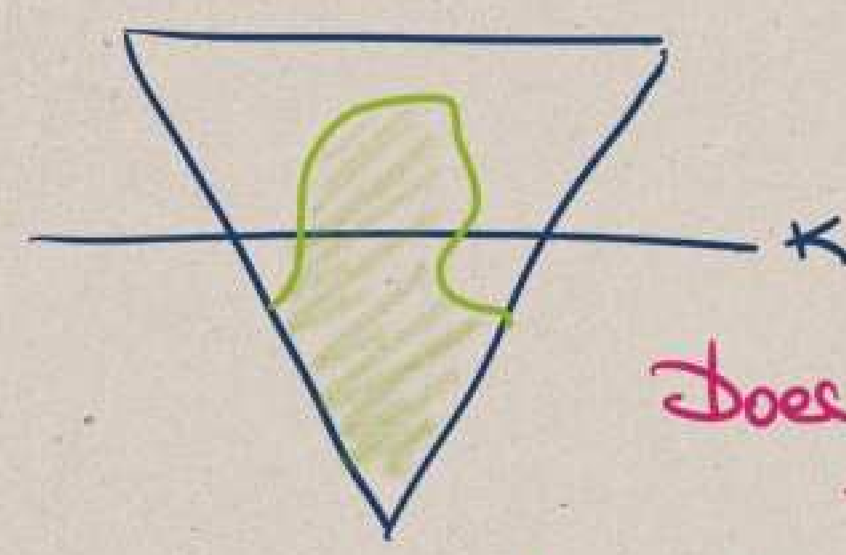
Answer: NO! Going beyond λ?

(8)

(8) We can show using (4) that

$$M \subseteq V_\lambda$$

[Example sheet #2]



Does it now look like this?

Not reaching up to λ, missing points below κ.

Answer: No

$\longrightarrow$ Next Lecture XI.