VIII EIGHTH LECTURE
LARGE CARDINALS

15 February 2023

GOAL

κ measurable

✓ $\Downarrow \qquad \not\Uparrow$ — Lecture VIII

κ weakly compact

Lecture VII → $\Downarrow \qquad \not\Uparrow$ — Lectures VIII & IX

κ inaccessible

For meas. $\Longrightarrow$ w.c., we need the method of ultraproducts.

Ultraproducts

Let κ be a cardinal, U ultrafilter on κ.
$\mathcal{M}_\alpha$ (for $\alpha < \kappa$) are $\mathcal{L}_S$-structure.
Define functions, relations, constants on
$\prod_{\alpha<\kappa} M_\alpha$ pointwise.

Let $M := \prod_{\alpha<\kappa} M_\alpha / U$ and write $[f]$ for U-eq. class of f.

Then $F^{\mathcal{M}}([f_0], \ldots, [f_n]) = [g]$

iff

$\{\alpha ; F^{\mathcal{M}_\alpha}(f_0(\alpha), \ldots, f_n(\alpha)) = g(\alpha)\} \in U$

and

$R^{\mathcal{M}}([f_0], \ldots, [f_n])$ iff

$\{\alpha ; R(f_0(\alpha), \ldots, f_n(\alpha))\} \in U$.

If φ is an FOL formula,

Łoś's Theorem says

$$\mathfrak{M} \models \varphi([f_0], \dots, [f_n]) \iff$$

$$\{\alpha ;\ \mathfrak{M}_\alpha \models \varphi(f_0(\alpha), \dots, f_n(\alpha))\} \in U.$$

We need Łoś's theorem for infinitary languages L_δ.

For this, let us analyze what $\varphi \in L_\delta$ looks like. A $\varphi \in L_\delta$ is a sequence of symbols, possibly transfinite.

Define length by recursion:

For FO formulas, this is just # of symbols, i.e., a natural number.

Note that atomic L_δ-formulas are FO formulas

$$lh(\exists^\alpha \vec{v} \varphi) := 1 + \alpha + lh(\varphi)$$

$$lh\left(\bigwedge_{\xi < \alpha} \varphi_\alpha\right) := 1 + \sum_{\xi < \alpha} lh(\varphi_\xi)$$

ORDINAL SUM

Prop. If $\varphi \in L_S$ and κ is regular, L_S is $\mathcal{L}_{\kappa\kappa}$-language, then
$$\mathrm{lh}(\varphi) < \kappa.$$

Pf. Since κ is cardinal, it's enough to show $|\varphi| < \kappa$.

We can easily prove this by induction, observing that the recursive definition steps of lh all preserve $|\varphi| < \kappa$.

[In the case of $\bigwedge_{\xi<\alpha}$, this uses the regularity of κ.].

q.e.d.

Corollary If κ is inaccessible & $|S| \leq \kappa$, then $|L_S| = \kappa$.

Pf. The set of all symbols has size κ in this case.

Via bijective encoding, a formula is just an element of $\kappa^{<\kappa} = \bigcup_{\lambda<\kappa} \kappa^\lambda$

If κ is inaccessible, then $|\kappa^{<\kappa}| = \kappa$. q.e.d.

COMMENT on $|\varphi|$

A formula is a function $\varphi : \mathrm{lh}(\varphi) \to \Sigma$ where Σ is the set of all symbols of L_S. Thus $|\varphi|$ is just the cardinality of that function, i.e. $|\varphi| = |\mathrm{dom}(\varphi)|$.

Theorem **Łoś for $\mathcal{L}_{\kappa\kappa}$-languages**

Suppose U is a κ-complete ultrafilter on κ. Then for every formula $\varphi \in L_S$, we have

$$\mathfrak{M} \models \varphi([\vec{f}])$$
$$\Longleftrightarrow$$
$$\{\alpha ; \mathfrak{M}_\alpha \models \varphi(\overrightarrow{f(\alpha)})\} \in U.$$

NOTATION

if $\vec{f} = (f_\xi ; \xi < \gamma)$ is a sequence of elts of $\prod_{\alpha<\kappa} M_\alpha$, I write $[\vec{f}]$ for $([f_\xi] ; \xi < \gamma)$ and $\overrightarrow{f(\alpha)}$ for $(f_\xi(\alpha) ; \xi < \gamma)$

Proof. As usual, induction on the complexity of formulas.

1. Atomic fmlas are just FO formulas, so it's the classical case.
2. Propositional connective work as in the classical case.
   [Note: $\neg$ uses "ultra".]

Left to deal with: $\exists^\alpha \vec{v}$; $\bigwedge_{\xi<\alpha}$.

$\underline{\exists^\alpha \vec{v}}$: $\longrightarrow$ ES#2

Proof is not really affected by the transfinite nature of the formula. [Note: Uses AC.!].

$\underline{\bigwedge_{\xi<\alpha}}$: $$\mathcal{M} \models \bigwedge_{\xi<\alpha} \varphi_\xi([\vec{f}])$$

$$\iff \text{for all } \xi<\alpha \quad \mathcal{M} \models \varphi_\xi([\vec{f}])$$

$$\overset{IH}{\iff} \text{for all } \xi<\alpha,\ X_\xi := \{\alpha ; \mathcal{M}_\alpha \models \varphi_\xi(\overrightarrow{f(\alpha)})\} \in U$$

Note: $X_\xi \supseteq X$, so if $X \in U$, then $X_\xi \in U$ $\Uparrow$

$\Downarrow$ Note: This direction uses κ-completeness $X = \bigcap_{\xi<\alpha} X_\xi$, so if $X_\xi \in U \Rightarrow X \in U$

$$X := \{\gamma ; \text{for all } \xi<\alpha\ \mathcal{M}_\gamma \models \varphi_\xi(\overrightarrow{f(\gamma)})\} \in U$$

$$\iff \{\gamma ; \mathcal{M}_\gamma \models \bigwedge_{\xi<\alpha} \varphi_\xi(\overrightarrow{f(\gamma)})\} \in U$$

q.e.d.

<u>Thm</u> If κ is measurable, then κ is weakly compact.

<u>Pf.</u> Suppose $|S| \le \kappa$, $\Phi \subseteq L_S$,
by earlier analysis, write $\Phi = \{\varphi_\alpha ; \alpha < \kappa\}$

$$\Phi_\mu := \{\varphi_\alpha ; \alpha < \mu\}$$

(*) Φ is κ-satisfiable $\Longleftrightarrow$ $\forall \mu < \kappa$ Φ_μ is satisfiable.

So, we need to show that if (*), then Φ is satisfiable.

Choose $\mathfrak{M}_\mu \models \Phi_\mu$. Let U be κ-complete nonprincipal uf. on κ and form

$$\mathfrak{M} := \prod_{\mu<\kappa} \mathfrak{M}_\mu / U .$$ the ultraproduct of the $\mathfrak{M}_\mu$ w.r.t. U

<u>Claim</u> $\mathfrak{M} \models \Phi$.

Since U is nonprincipal κ-complete, we saw that $X \in U \longrightarrow |X| = \kappa$. Lemma on p. 7 Lecture VI

So for each μ, $\{\alpha ; \alpha > \mu\} \in U$.

For each $\gamma < \kappa$;

$$\{\alpha ; \alpha > \gamma\} \in U \Longrightarrow \{\alpha ; \mathfrak{M}_\alpha \models \varphi_\gamma\} \in U \xRightarrow{\text{Łoś}} \mathfrak{M} \models \varphi_\gamma .$$

Since γ was arbitrary, $\mathfrak{M} \models \Phi$. q.e.d.

Now: κ inacc. $\not\Rightarrow$ κ weakly compact. (+)

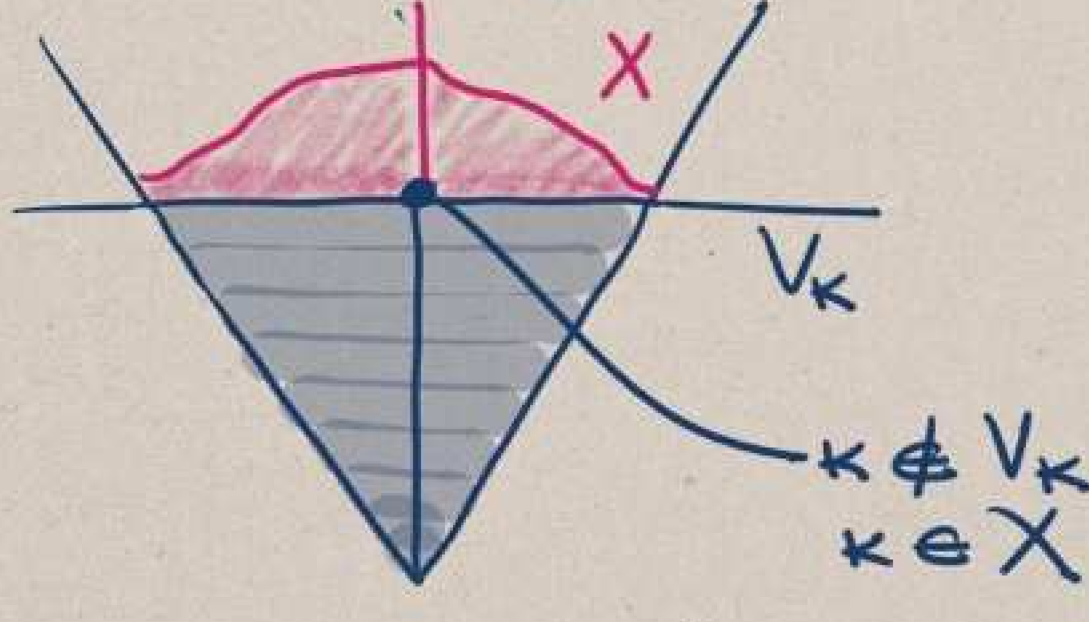
Def. κ has the KEISLER extension (KEP) property if there is a transitive set $X \supsetneq V_\kappa$ s.t. $\kappa \in X$ s.t. $V_\kappa \preccurlyeq X$.

We'll show that if κ is weakly compact, then κ has the KEP.

Let us derive (+) from the KEP.

Thm If κ is weakly compact, then there is $\lambda < \kappa$ s.t. λ is inaccessible.

Cor In particular, the least inaccessible cannot be weakly compact.



Note: "inaccessible" is absolute between V_α and V $\ddagger$ p. 7 of Lecture III

& downwards absolute between X and V $\star$ p. 2-4 of Lecture VI

In the following, let X be such that $V_\kappa \preccurlyeq X$ as in KEP!

$$V \models \kappa \text{ is weakly compact}$$

$$\Longrightarrow V \models \kappa \text{ inaccessible} \quad [\text{Lecture VII}]$$

$$\overset{\star}{\Longrightarrow} X \models \kappa \text{ inaccessible}$$

$$\Longrightarrow X \models \mathrm{IC}$$

$$\overset{V_\kappa \preccurlyeq X}{\Longrightarrow} V_\kappa \models \mathrm{IC}$$

$$\Longrightarrow V_\kappa \models \exists \lambda \ \lambda \text{ inaccessible}$$

$$\overset{\ddagger}{\Longrightarrow} V \models \exists \lambda \ (\lambda < \kappa \ \& \ \lambda \text{ inaccessible}).$$

q.e.d.