# VII SEVENTH LECTURE
# LARGE CARDINALS

13 February 2023

Measurable cardinals:

$\kappa$ measurable iff there is a nonprincipal $\kappa$-complete ultrafilter on $\kappa$

We proved (Lecture $\underline{\text{VI}}$):

Every measurable cardinal is inaccessible.

Today: Another LCA: WEAKLY COMPACT CARDINALS

# Weakly compact cardinals

## INFINITARY LANGUAGES

A language $L_S$ is an $\mathcal{L}_{\kappa\kappa}$-language if it has

(a) $\kappa$ many variables
(b) a set $S$ of standard non-logical symbols as in FOL (i.e., finite arity)
(c) infinitary symbols $\bigwedge_{\xi<\alpha}$ and $\bigvee_{\xi<\alpha}$ for any $\alpha<\kappa$
(d) infinitary quantifier symbols $\exists^\alpha$ and $\forall^\alpha$ for $\alpha<\kappa$

SYNTAX Terms & atomic formulas are defined as usual.

If $\varphi_\xi$ for $\xi<\alpha$ is an $L_S$-formula, then so is $\bigwedge_{\xi<\alpha}\varphi_\xi$ and $\bigvee_{\xi<\alpha}\varphi_\xi$.

Remark. These can have infinite length and also $\infty$ly many free variables.

If $\vec{v} := (v_\xi \,;\, \xi < \alpha)$ is a sequence of variables, and $\varphi$ is an $\mathcal{L}_S$-formula, then so are

$$\exists^\alpha \vec{v}\,\varphi \quad \text{and} \quad \forall^\alpha \vec{v}\,\varphi.$$

<u>SEMANTICS</u> Just define the new cases:

$$\mathcal{M} \models \bigwedge_{\xi<\alpha} \varphi_\xi \;:\Longleftrightarrow\; \text{for all } \xi < \alpha, \; \mathcal{M} \models \varphi_\xi .$$

$$\mathcal{M} \models \exists^\alpha \vec{v}\,\varphi \;:\Longleftrightarrow\; \text{there is a function } a : \alpha \longrightarrow M \text{ s.t.}$$

$$\mathcal{M}\frac{a}{\vec{v}} \models \varphi.$$

$v_\xi$ is interpreted by $a(\xi)$

These languages are called

INFINITARY LANGUAGES

# Infinitary languages go beyond FOL

1. $$\forall^{\omega} \vec{x} \bigvee_{i \neq j} x_i = x_j \iff: \Phi_{Fin}$$

Clearly, $\mathfrak{M} \models \Phi_{Fin}$ iff $M$ is finite.

$\Phi_{Fin}$ is an $\mathcal{L}_{\omega_1 \omega_1}$-formula.

2. Suppose there are countably many constant symbols $c_n$

$$\Phi_{\mathbb{N}} :\iff \forall x \bigvee_{n \in \mathbb{N}} x = c_n \wedge \bigwedge_{n \neq m} c_n \neq c_m$$

This is also $\mathcal{L}_{\omega_1 \omega_1}$

Then $\mathfrak{M} \models \Phi_{\mathbb{N}}$ iff $|M| = \aleph_0$.

<u>Comment</u>. For FOL, these are properties that cannot be axiomatised, as usually proved by compactness.

Thus: Ordinary compactness **must** fail for $\mathcal{L}_{\kappa\kappa}$-languages.

**Def.** A set of formulas $\Phi \subseteq L_S$ is **$\lambda$-satisfiable** if for all $\Phi_0 \subseteq \Phi$ with $|\Phi_0| < \lambda$, $\Phi_0$ is satisfiable.

[Thus: "finitely satisfiable" $\Longleftrightarrow$ $\aleph_0$-satisfiable]

**Def.** A cardinal $\kappa$ is called **weakly compact** if for all $\mathcal{L}_{\kappa\kappa}$-languages $L_S$ with $|S| \leq \kappa$ and all $\Phi \subseteq L_S$,

$$\Phi \ \kappa\text{-satisfiable} \implies \Phi \text{ satisfiable.}$$

Remarks
1. If $\kappa = \omega$, this is just the compactness theorem, so $\omega$ is weakly compact.
2. There is a notion of **strong compactness** which drops the condition $|S| \leq \kappa$.
   $\longrightarrow$ ES#2.

**Theorem** Every weakly compact cardinal is inaccessible.

**Proof.** **REGULAR**

Suppose $\kappa = \bigcup X$ where $X \subseteq \kappa$ with $|X| < \kappa$.
Consider $L_S$ where $S = \{c_\alpha ; \alpha < \kappa\} \cup \{c\}$
Clearly $|S| = \kappa$, so we can apply weak compactness to $L_S$.

$$\varphi := \bigvee_{\beta \in X} \bigvee_{\alpha < \beta} c = c_\alpha .$$

$|X| < \kappa$ $|\beta| < \kappa$, so $\varphi$ is an $\mathcal{L}_{\kappa\kappa}$-formula

$$\Phi = \{\varphi\} \cup \{c_\alpha \neq c ; \alpha < \kappa\}$$

Clearly, $\Phi$ is inconsistent, but if $\Phi_0 \subseteq \Phi$ with $|\Phi_0| < \kappa$, then there is $\alpha$ s.t. $c_\alpha \neq c \notin \Phi_0$, so we can interpret $c$ by whatever interprets $c_\alpha$.
So $\Phi$ is $\kappa$-satisfiable; contradiction!

## STRONG LIMIT

Suppose $2^\lambda \geq \kappa$ for some $\lambda < \kappa$. Let us use the following constants

So, this defines a function $g : \lambda \longrightarrow 2$ by $g(\alpha) = 0 \iff \mathfrak{M} \models c_\alpha = c_\alpha^0$.

$c_\alpha, c_\alpha^0, c_\alpha^1$ for $\alpha < \lambda$.

[ $|S| = \lambda < \kappa$, so this is a valid language ]

$$\varphi := \bigwedge_{\alpha<\lambda} c_\alpha^0 \neq c_\alpha^1 \wedge \bigwedge_{\alpha<\lambda} c_\alpha = c_\alpha^0 \vee c_\alpha = c_\alpha^1$$

Let $f : \lambda \longrightarrow 2$ be arbitrary and define

$$\varphi_f := \bigvee_{\alpha<\lambda} c_\alpha \neq c_\alpha^{f(\alpha)}$$

[ The function defined by the $c_\alpha$'s is different from $f$. ]

$$\Phi := \{\varphi\} \cup \{\varphi_f \,;\, f \in 2^\lambda\}$$

$\varphi$ says that the $c_\alpha$'s define a function, but the $\varphi_f$'s say that it is not equal to a given one, so $\Phi$ is inconsistent.

If $\Phi_0 \subseteq \Phi$ with $|\Phi_0| < \kappa \leq 2^\lambda$, then there is $f \in 2^\lambda$ s.t. $\varphi_f \notin \Phi_0$, so we can just let $c_\alpha = c_\alpha^{f(\alpha)}$.

Thus $\Phi$ is $\kappa$-satisfiable; contradiction!

q.e.d.

## MAJOR GOAL:

MC $\kappa$ measurable

$\Downarrow \quad \not\Uparrow$

WC $\kappa$ weakly compact

$\Downarrow \quad \not\Uparrow$

IC $\kappa$ inaccessible

We obtain

$$\mathrm{IC} <_{\mathrm{Cons}} \mathrm{WC} <_{\mathrm{Cons}} \mathrm{MC}.$$

The LINEARITY PHENOMENON of the large cardinal hierarchy:

Usually, given two LCA $\Phi C$, $\Psi C$, it'll turn out that

either $\Phi C \equiv_{\mathrm{Cons}} \Psi C$

or $\Phi C <_{\mathrm{Cons}} \Psi C$

or $\Psi C <_{\mathrm{Cons}} \Phi C$.

MC: measure theory
WC: logic
IC: cardinals

This linearity is unexpected, given that the motivation for the LCA is from entirely different areas of maths!

This allows us the measure the logical strength of any interesting axiom candidate $\varphi$ in the consistency strength hierarchy by comparing it with the Large Cardinal hierarchy as a measuring rod.

First subgoal: $\kappa$ measurable

$$\Longrightarrow$$

$\kappa$ weakly compact.

[Uses the technique of ultraproducts.]

If $\mathcal{M}_\alpha$ is an $\mathcal{L}_S$-structure for $\alpha<\kappa$, then we define the product

$\mathcal{M} := \prod_{\alpha<\kappa} \mathcal{M}_\alpha$ by letting $M = \prod_{\alpha<\kappa} M_\alpha$ and by defining functions, constants, and relations pointwise.

E.g. $h^{\mathcal{M}}(f_0,\dots,f_n)(\alpha) := h^{\mathcal{M}_\alpha}(f_0(\alpha),\dots,f_n(\alpha))$

If $F$ is a filter on $\kappa$, then

$$f \sim_F g :\Longleftrightarrow \{\alpha;\, f(\alpha)=g(\alpha)\} \in F$$

is an equivalence relation. Write $[f]_F = [f]$ for eq. classes.

Then:

$\mathcal{M}/F$ is the model consisting of the $\sim_F$-equivalence relations; check that the operations are well-defined

$\longrightarrow$ the REDUCED PRODUCT

Known from Model Theory:

Łoś's Theorem.

We need Łoś's Theorem for $\mathcal{L}_{\kappa\kappa}$-languages

$\longrightarrow$ NEXT TIME !