LARGE CARDINALS

Sixth Lecture: 8 February 2023

CORRECTION:

Lecture V
Page 7

Def. QF is the smallest class of fmlas containing the atomic formulas & closed under propositional connectives.

Δ_0 is the smallest class containing atomic fmlas & closed under prop. conn. & bdd. quantification.

Σ_1 is the smallest class containing Δ_0 & closed under ex. quantification ($\varphi \in \Sigma_1 \longrightarrow \exists x \varphi \in \Sigma_1$)

Π_1 is the smallest class containing Δ_0 & closed under univ. qf. ($\varphi \in \Pi_1 \longrightarrow \forall x \varphi \in \Pi_1$).

This is corrected from the lecture. In the lecture, we had the closure of QF under bdd quantification. Check that this is a different class.

If you close QF under bdd. quantification, you get a SMALLER class of formulas than Δ_0.

RECAP

Thm All Δ_0^T formulas are absolute between transitive models of T.

Note. In the proof $M \models \exists x \in y\, \varphi \iff N \models \exists x \in y\, \varphi$, the direction "$\Longrightarrow$" was considerably easier and didn't use the assumptions.

From Lecture V
Page 10.

To show: $M \models \exists x(x\in a \wedge \varphi) \iff N \models \exists x(x\in a \wedge \varphi)$

"$\Longrightarrow$" $\quad M \models \exists x(x\in a \wedge \varphi)$

$\iff$ ex. $b\in M$ $\quad M\models b\in a \wedge \varphi$

$\iff$ ex. $b\in M$ $\quad M\models b\in a$ and $M\models \varphi$

$\overset{\text{IH}}{\iff}$ ex. $b\in M$ $\quad N\models b\in a$ and $N\models\varphi$

$\iff$ ex. $b\in M$ $\quad N\models b\in a\wedge\varphi$

$\overset{M\subseteq N}{\Longrightarrow}$ ex. $b\in N$ $\quad N\models b\in a\wedge\varphi$

$\iff$ $N\models \exists x(x\in a\wedge\varphi)$.

This direction did not need transitivity or the role of the boundedness of the quantifier!

"$\Longleftarrow$" $\quad N\models\exists x(x\in a\wedge\varphi)$

$\iff$ ex. $b\in N$ $\quad N\models b\in a\wedge\varphi$

$\iff$ ex. $b\in N$ $\quad N\models b\in a$ and $N\models\varphi$

We have $\left.\begin{array}{l} b\in N \\ a\in M \\ b\in a\end{array}\right\} \overset{}{\underset{\text{Transitivity}}{\Longrightarrow}} b\in M$

$\Longrightarrow$ ex. $b\in M$ $\quad N\models b\in a$ and $N\models\varphi$

$\overset{\text{IH}}{\iff}$ ex. $b\in M$ $\quad M\models b\in a\wedge\varphi$

$\iff$ $M\models\exists x(x\in a\wedge\varphi)$.

q.e.d.

Proof analysis gives:

If φ is absolute between M & N, then $\exists x\varphi$ is upwards abs. between M & N.

Corollary Σ_1 formulas are upwards absolute for trs models

Π_1 fmlas are downwards abs. for trs models

APPLICATION Important fmlas are $\Delta_0, \Sigma_1, \Pi_1$.

Ex 1 "being empty" [Lecture V] is Δ_0

Ex. 2 "x is a subset of y" $x\subseteq y \iff \forall z(z\in x \to z\in y)$ is Δ_0

Ex. 3 [ES#1] "x is a function" "x is onto z"

"$x: y\to z$" "x is a bijection"

"x is injection" "$x\subseteq y$ is cofinal"

are Δ_0

Ex. 4 "x is a cardinal"

"x is regular"

"x is strong limit"

are all Π_1^{ZFC}

[assuming that being an ordinal is Δ_0^{ZFC}]

What about "x is an ordinal"?

Official definition

x is a transitive set s.t. $(x, \in)$ is a wellorder

BUT: "wellfoundness" is not Δ_0: it has an unbounded quantifier

Solution: Axiom of Foundation implies that this is eq. to "x is a transitive set s.t. $(x, \in)$ is a total order"

Thus it is Δ_0^{ZFC}

E.g., transitivity is

$\forall a \in x \, \forall b \in x \, \forall c \in x \, (a \in b \wedge b \in c \Longrightarrow a \in c)$

i.e. Δ_0

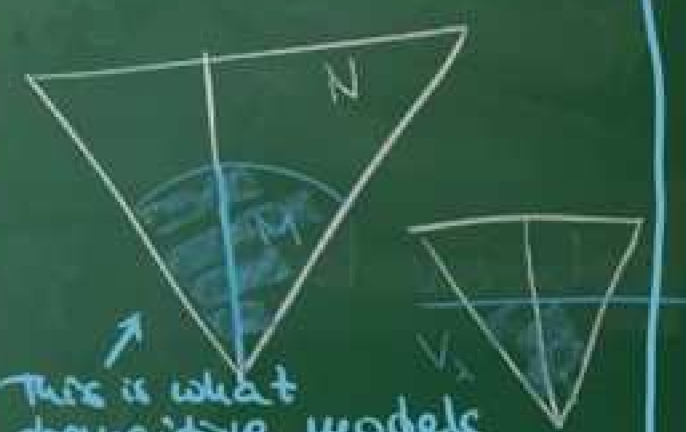
Thus if $M \subseteq N$ is transitive, then

$$\mathrm{Ord}^M \subseteq \mathrm{Ord}^N.$$

In general, we do not have $=$:

e.g., if $V_\lambda \subseteq V_\kappa$ are both models of ZFC

$\mathrm{Ord}^{V_\lambda} = \lambda \neq \kappa = \mathrm{Ord}^{V_\kappa}$



Remark Since "is an ordinal" is only Δ_0^{ZFC}, this is inherited by "is a cardinal" etc.

__Also__

If $\kappa \in M$, $N \models \kappa$ is a cardinal,

then by Π_1, $M \models \kappa$ is a cardinal

__But__ It's possible that

$$\mathrm{Ord} \cap M = \alpha < \omega_1^N$$

Mostowski collapse of $M \preceq N$ ctbl

so there may be no instances of this

__Def__ $M \subseteq N$ is called an __inner model__ if it's transitive & $\mathrm{Ord}^M = \mathrm{Ord}^N$.

In this case,

$$\mathrm{Card}^M \supseteq \mathrm{Card}^N$$

$$\mathrm{Inacc}^M \supseteq \mathrm{Inacc}^N$$

This is what inner models in general look like.

MEASURABLE CARDINALS

Def If X is a set, we call $F \subseteq \mathcal{P}(X)$ a **filter** if

Collection of large sets

(i) $\emptyset \notin F$, $X \in F$

(ii) $A, B \in F \Rightarrow A \cap B \in F$

(iii) $A \in F$, $B \supseteq A \Rightarrow B \in F$

F is called **ultra**

(iv) for all A, either $A \in F$ or $X \setminus A \in F$

Let λ be a cardinal, F is called **λ-complete** if for all $\{A_\alpha ; \alpha < \gamma\} \subseteq F$ where $\gamma < \lambda$

we have $\bigcap_{\alpha<\gamma} A_\alpha \in F$

Elements of the filter are the **large** subsets of X.

Note (ii) implies that every filter is $\aleph_0$-complete.

Example If $x \in X$, define $U_x := \{A \subseteq X ; x \in A\}$

This is a λ-complete uf for all λ

Called the principal ultrafilter generated by x

If F is an ultrafilter s.t. for all $x \in X$, $\{x\} \notin F$, then F is called non-principal

Using AC (Zorn's Lemma), we can show that n-p uf exist

Lemma If U is n-p uf, λ-complete

$|A| < \lambda \implies A \notin U$.

Pf $|A| < \lambda$, then $A = \bigcup_{a \in A} \{a\}$, so it's a union of $<\lambda$ sets $\notin U$.

$$X \setminus A = \bigcap_{a \in A} \underbrace{X \setminus \{a\}}_{\in U \text{ since } U \text{ is n-p \& ultra}} \in U \quad [\text{by } \lambda\text{-completeness}]$$

So $A \notin U$. q.e.d

Consider $F := \{A \subseteq \mathbb{N} ; \mathbb{N} \setminus A \text{ is finite}\}$.
This is known as the Fréchet filter. It is NOT an ultrafilter.

If $U \supseteq F$ is an ultrafilter, it cannot be principal (Why?).

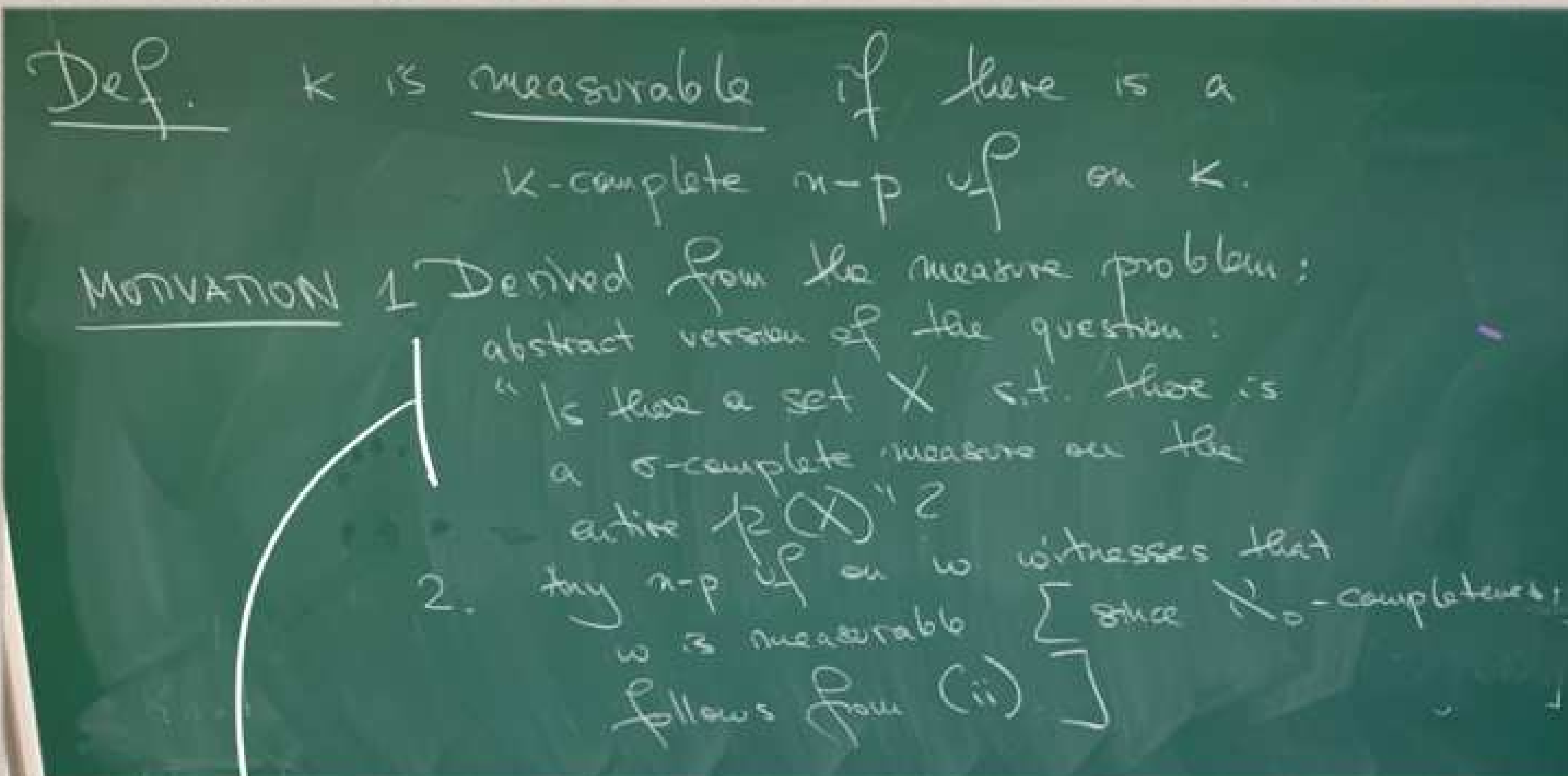
Observe that a filter U is maximal (i.e., if $F' \supseteq U$ is a filter, then $F' = U$) if and only if U is ultra. (Why?)

Use Zorn's Lemma to find a maximal filter extending F.

This argument shows:

if U is an ultrafilter, then
U is λ-complete iff for any $\{A_\alpha ; \alpha < \gamma\}$
$\gamma < \lambda$ s.t.
$A_\alpha \notin U$, we
have $\bigcup_{\alpha<\gamma} A_\alpha \notin U$

This is not in general true for filters.



The measure problem:

Is there a σ-additive $\mu : \mathcal{P}([0,1]) \to [0,1]$
s.t. $\mu(\emptyset) = 0$ and $\mu([0,1]) = 1$.

σ-additivity corresponds to $\aleph_1$-completeness

σ-additivity implies:

if $\mu(A_n) = 0 \implies \mu(\bigcup_{n \in \mathbb{N}} A_n) = 0$

[compare remark on p. 7]

Note: If there is a κ with an $\aleph_1$-complete non-principal uf, then there is a λ with a λ-complete non-principal uf.

Theorem Every uncountable measurable cardinal is inaccessible.

Proof **Regular** Suppose $C \subseteq \kappa$ is cofinal with $|C| < \kappa$.
If $\gamma \in C$, $\gamma \subseteq \kappa$. Cofinality means $\kappa = \bigcup_{\gamma \in C} \gamma$
with $|\gamma| < \kappa$, so by Lemma $\gamma \notin U$.
Thus $\kappa \notin U$ in contradiction to (.).

Strong limit Suppose there is $\lambda < \kappa$, s.t. $2^\lambda \geq \kappa$.
So, there is $X \subseteq \{f : \lambda \longrightarrow 2\}$ that has cardinality κ: write $X = \{f_\alpha, \alpha < \kappa\}$ no repetitions

For $\gamma < \lambda$, consider $Z_\gamma = \{\alpha ; f_\alpha(\gamma) = 0\}$ Define $i : \lambda \longrightarrow 2$
Thus for each γ either $Z_\gamma \in U$ or $\kappa \setminus Z_\gamma \in U$. $i(\gamma) = \begin{cases} 0 & \text{if } Z_\gamma \in U \\ 1 & \text{if } Z_\gamma \notin U \end{cases}$

$Z := \bigcap_{\gamma < \lambda} H_\gamma \in U$ by κ-completeness

If $\alpha \in Z$, then $f_\alpha = i$. where $f_\alpha = i$
$\Longrightarrow Z \subseteq \{\alpha\}$; Contradiction to non-principality. q.e.d.

Say $H_\gamma := \begin{cases} Z_\gamma & \text{if } Z_\gamma \in U \\ \kappa \setminus Z_\gamma & \text{if } Z_\gamma \notin U \end{cases}$

Suppose $\alpha \in Z$. Then $\alpha \in H_\gamma$ for all $\gamma < \lambda$.
I.e., $f_\alpha(\gamma) = i(\gamma)$ for all $\gamma < \lambda$
So $f_\alpha = i$.
Thus, if $\alpha \in Z$, then f_α is uniquely determined,
so Z is either $\emptyset$ or $\{\alpha\}$ with $f_\alpha = i$.