V

FIFTH LECTURE
6 FEBRUARY 2023

LARGE CARDINALS

ABSOLUTENESS FOR MODELS OF SET THEORY

Let $M \subseteq N$ be any $\mathcal{L}$-structures and φ be a formula. We say that

φ is absolute between M & N

if for all $x_0, \dots, x_n \in M$, we have

$$M \models \varphi(x_0, \dots, x_n) \Longleftrightarrow N \models \varphi(x_0, \dots, x_n)$$

So: $M \preccurlyeq N \Longleftrightarrow$ all φ are absolute between M & N

<u>Def.</u> φ is upwards absolute between M & N if "$\Longrightarrow$";

φ is downwards absolute between M & N if "$\Longleftarrow$".

From model theory:

$M \subseteq N$ is a substructure if all atomic formulas are absolute between M & N.

Example If $(H, 0, +, -)$ is a group and $G \subseteq H$ is a substructure of H, then $0 \in G$; if $x, y \in G$, then $x+y, -x \in G$.

Then $(G, 0, +, -)$ is a group.

However $\mathcal{L}_\in$ (the language of set theory) has no real structure: RELATIONAL LANGUAGE.

What about $\emptyset, \cup, \cap, \ldots$?

In set theory, these are all defined symbols and not part of the language.

$$x = \emptyset \iff \forall y \, (y \notin x).$$

Problem: This is not an atomic formula, but has quantifiers, so it's not in general preserved by substructures.

EXAMPLE Let $N \models ZFC$ and $M \subseteq N$ be $\{1\} = M$.

Consider the formula defining the empty set:

$$\varphi(x) \iff \forall y\,(y \notin x).$$

Clearly, $(N, \in) \models \neg\varphi(1)$.

But what is happening in M?

$$(M, \in) = (\{1\}, \in) \models \forall y\,(y \notin 1)$$
$$= \varphi(1).$$

Analysis We added 1 to M, but neglected the elements of 1.

In other words, $\{1\}$ is not a transitive set.

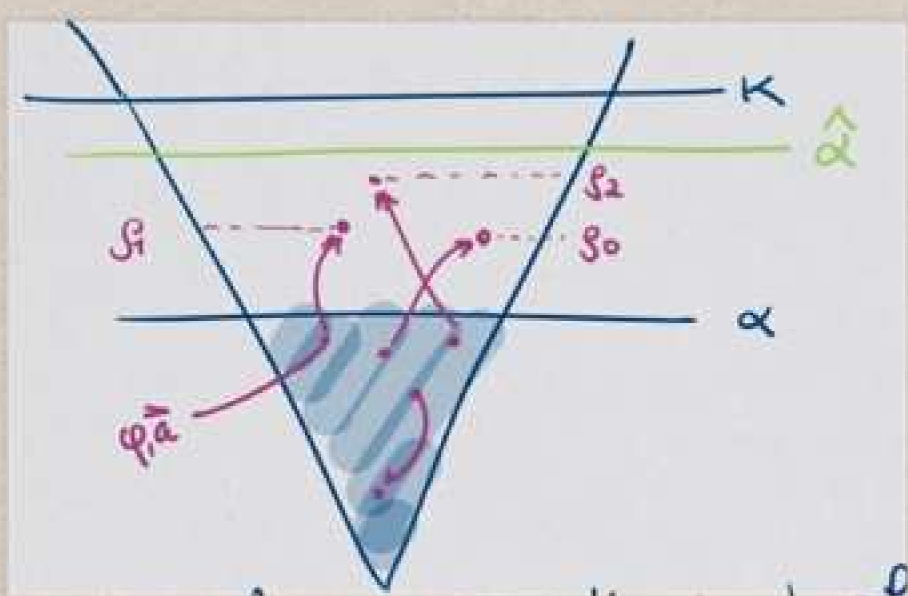
The problem disappears if M is a transitive set.

Def. If $M \subseteq N$, then we say **M is transitive in N** if whenever

$$\left.\begin{array}{l} x, y \in N \\ x \in M \\ y \in x \end{array}\right\} \Longrightarrow y \in M.$$

BRIEF DIGRESSION ON NONTRANSITIVE MODELS

A version of the Löwenheim-Skolem theorem says: for any N there is a ctble M s.t. $M \preccurlyeq N$.



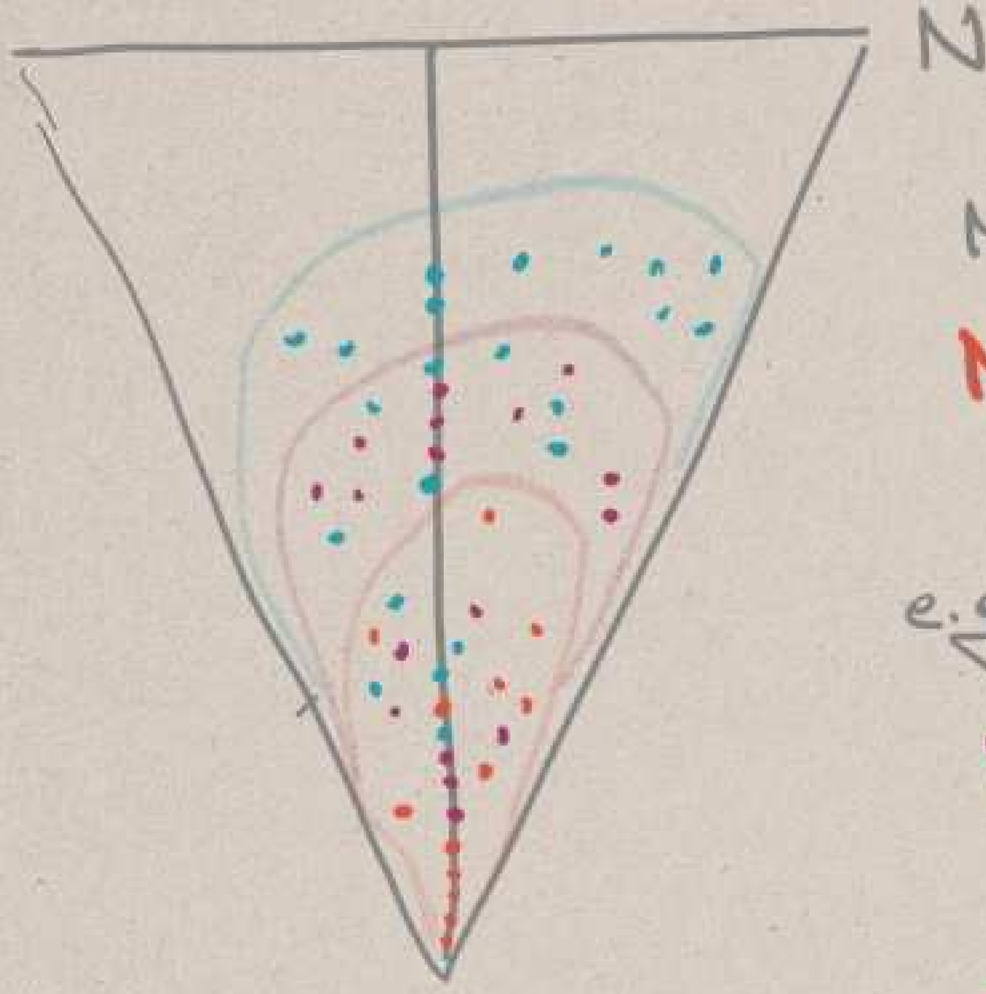
From Lecture IV, page 8.

[Do the argument from Lecture IV but instead of adding all elements of V_{α_i} in the i-th step, only add the witnesses themselves. These are only ctbly many.]

The union of all of those is a ctble union of ctble sets, $\preccurlyeq$ by TVT.

This is NOT going to be transitive!

The fmla "there is a least uncountable cardinal" has a unique witness ω_1^N. This is added to M in the very first stage. But ω_1^N is uncountable, so $\omega_1^N \not\subseteq M$. So M is not transitive.



$M_0 := \emptyset$

M_1 := witnesses to "all" fmlas w/ parameters in M_0

e.g. $\exists x\, (x = \aleph_1)$

M_2 := witnesses to a fmlas w/ parameters in M_1

M_3 := witnesses to a fmlas w/ param. in M_2

The whole structure is very gappy. It contains all elements of $\mathbb{N}$ [since those are definable], ω itself [definable], $\omega_1, \omega_2, \omega_3, \ldots$ [all definable], but leaves **HUGE** spaces between them.

$$\sup\{\alpha \in M;\ \alpha < \omega_1^N\} < \omega_1^N.$$

Reminder.

The analogue of 'subset collapse' is:

Theorem 4 (Mostowski's Collapsing Theorem). Let r be a relation on a set a that is well-founded and extensional. Then there exists a transitive set b, and a bijection $f : a \to b$ such that $(\forall x, y \in a)(x\ r\ y \Leftrightarrow f(x) \in f(y))$. Moreover, b and f are unique.

Remark. 'Well-founded' and 'extensional' are trivially necessary.

From Leader's L&ST notes (page 40).

So in this case, we have

countable $\rightarrow M \preccurlyeq N$

If $N \models ZFC$, then so is M.
So $\in$ is extensional on M.

$\xRightarrow{\text{MOSTOWSKI}}$ there is an iso $f : M \to T$ where T is transitive:

$$(M, \in) \cong (T, \in)$$

This is a case of an elementary embedding: the inverse of the Mostowski collapse.

In particular, $(T, \in) \models ZFC$.
and $|M| = |T| = \aleph_0$.

This cannot be an elementary substructure, since $T \cap \mathrm{Ord} = \alpha < \omega_1$, but $T \models \aleph_1$ exists.

Thus, let β be s.t. $T \models \beta = \omega_1$, then

$$N \not\models \beta = \omega_1.$$

Def. Let Λ be a class of formulas.
We say
Λ is closed under propositional connectives
if $\varphi, \psi \in \Lambda$, then $\varphi \wedge \psi, \varphi \vee \psi, \neg\varphi \in \Lambda$
Λ is closed under bounded quantification
if $\varphi \in \Lambda$, then $\exists x (x \in y \wedge \varphi) \in \Lambda$

BOUND
$\exists x \in y\, \varphi$

Def. QF is the smallest class of fmlas containing the atomic formulas & closed under propositional connectives.
Δ_0 is the smallest class containing atomic fmlas & closed under prop. conn. & bdd. quantification
Σ_1 is the smallest class containing Δ_0 & closed under ex. quantification
$(\varphi \in \Sigma_1 \longrightarrow \exists x \varphi \in \Sigma_1)$
Π_1 is the smallest class containing Δ_0 & closed under univ. qf.
$(\varphi \in \Pi_1 \longrightarrow \forall x \varphi \in \Pi_1)$.

This is corrected from the lecture. In the lecture, we had the closure of QF under bdd. quantification. Check that this is a different class.

If T is any theory, we write

Δ_0^T

Σ_1^T

Π_1^T

for the class of formulas φ s.t. there is a $\Delta_0 / \Sigma_1 / \Pi_1$ formula ψ and $T \vdash \varphi \leftrightarrow \psi$.

This avoids the problem that $\Delta_0, \Sigma_1, \Pi_1$ are purely syntactically defined.

<u>Example</u>

$$\begin{aligned}
\varphi(x) &:\Longleftrightarrow \forall y\,(y \notin x). \\
&\Longleftrightarrow \neg \exists y \neg (y \notin x) \\
&\Longleftrightarrow \neg \exists y\,(y \in x) \\
&\Longleftrightarrow \boxed{\neg \underbrace{\exists y\,(y \in x \land x = x)}_{\Delta_0 \text{ syntactically}}}
\end{aligned}$$

Thus φ is Δ_0^T for any T.

Theorem If M is transitive in N and $M, N \models T$, then all Δ_0^T formulas are absolute between M & N.

Proof. Since $M, N \models T$, we can w.l.o.g. assume that φ is a Δ_0 formula. Thus, we prove this by induction on the complexity of Δ_0 formulas.

So: need to show that

① atomic formulas are absolute [trivially true]

② if φ, ψ are absolute, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg \varphi$.
[Just ordinary semantics of $\wedge, \vee, \neg$.]

③ Bounded quantification.

Suppose φ absolute; consider $\exists x (x \in y \wedge \varphi)$
So fix $a \in M$ and show

$$M \models \exists x (x \in a \wedge \varphi) \iff N \models \exists x (x \in a \wedge \varphi).$$

To show: $M \models \exists x(x \in a \wedge \varphi) \Longleftrightarrow N \models \exists x(x \in a \wedge \varphi)$

"$\Longrightarrow$" $\quad M \models \exists x(x \in a \wedge \varphi)$

$\Longleftrightarrow$ ex. $b \in M$ $\; M \models b \in a \wedge \varphi$

$\Longleftrightarrow$ ex. $b \in M$ $\; M \models b \in a$ and $M \models \varphi$

$\overset{\text{IH}}{\Longleftrightarrow}$ ex. $b \in M$ $\; N \models b \in a$ and $N \models \varphi$

$\Longleftrightarrow$ ex. $b \in M$ $\; N \models b \in a \wedge \varphi$

$\overset{M \subseteq N}{\Longrightarrow}$ ex. $b \in N$ $\; N \models b \in a \wedge \varphi$

$\Longleftrightarrow$ $N \models \exists x(x \in a \wedge \varphi)$.

"$\Longleftarrow$" $\quad N \models \exists x(x \in a \wedge \varphi)$

$\Longleftrightarrow$ ex. $b \in N$ $\; N \models b \in a \wedge \varphi$

$\Longleftrightarrow$ ex. $b \in N$ $\; N \models b \in a$ and $N \models \varphi$

We have $\left.\begin{array}{l} b \in N \\ a \in M \\ b \in a \end{array}\right\} \overset{\text{Transitivity}}{\Longrightarrow} b \in M$

$\Longrightarrow$ ex. $b \in M$ $\; N \models b \in a$ and $N \models \varphi$

$\overset{\text{IH}}{\Longleftrightarrow}$ ex. $b \in M$ $\; M \models b \in a \wedge \varphi$

$\Longleftrightarrow$ $M \models \exists x(x \in a \wedge \varphi)$.

q.e.d.