IX LARGE CARDINALS

Ninth Lecture

20 February 2023

GOAL

κ measurable

✓ $\Downarrow$ $\not\Uparrow$ — Lecture VIII

κ weakly compact

Lecture VII → $\Downarrow$ $\not\Uparrow$ — Lectures VIII & IX

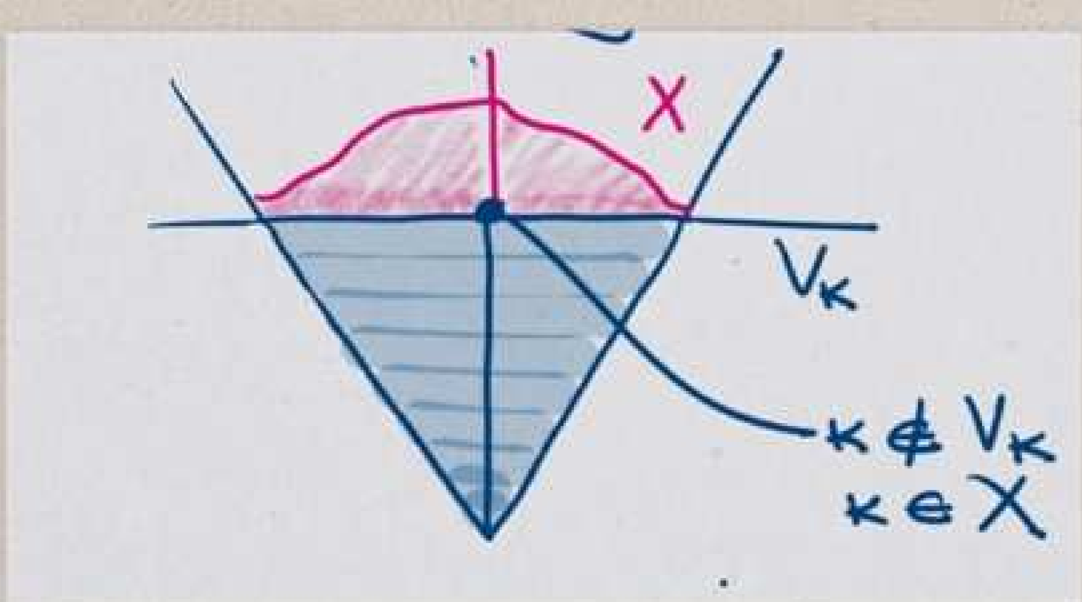
κ inaccessible

← We are in the proof of "inacc. $\not\Rightarrow$ weakly compact"

Def. κ has the KEISLER extension property (KEP) if there is a transitive set $X \supsetneq V_\kappa$ s.t. $\kappa \in X$ s.t. $V_\kappa \preccurlyeq X$.

FROM LECTURE VIII

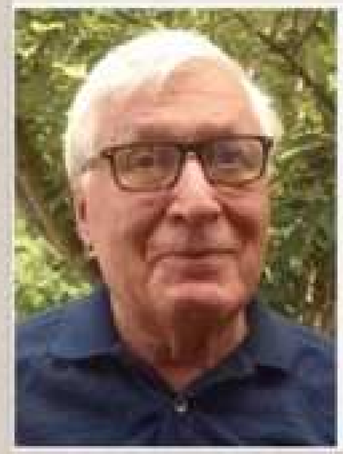
We proved: If κ is inaccessible and has the KEP, there is $\lambda < \kappa$ inaccessible.



Remains to prove

Theorem Every weakly compact cardinal has the KEP.

Def. κ has the KEISLER extension (KEP) property if there is a transitive set $X \supsetneq V_\kappa$ s.t. $\kappa \in X$ s.t. $V_\kappa \preccurlyeq X$.



JEROME KEISLER (born 1936)

Proof. Since κ is inaccessible, we have $|V_\kappa| = \kappa$.
Consider language L_κ, an $\mathcal{L}_{\kappa\kappa}$-language with additional constant symbol c_x for $x \in V_\kappa$.

$$\mathcal{V} := (V_\kappa, \in, \{x ; x \in V_\kappa\})$$

$$\Phi := \mathrm{Th}_{L_\kappa}(\mathcal{V}).$$

Observe:

1. Models of Φ are wellfounded.

$$\varphi := \forall^\omega \vec{v} \bigvee_{i \in \mathbb{N}} v_{i+1} \notin v_i .$$

φ characterises wellfoundedness

Clearly $\mathcal{V} \models \varphi$, and thus $\varphi \in \Phi$.

(2) If $\mathcal{M} \models \Phi$, & consider

$$W := \{ c_x^{\mathcal{M}} ; x \in V_\kappa \} \subseteq M$$

I claim that W is an isomorphic copy of V_κ sitting transitively inside M.

$$\psi_x := \forall z \left(z \in c_x \longleftrightarrow \bigvee_{y \in x} z = c_y \right)$$

since κ is inaccessible, $|x| < \kappa$.

As before $\mathcal{V} \models \psi_x$, so $\psi_x \in \Phi$.

This shows that W is transitive in M and

$x \longleftrightarrow c_x^{\mathcal{M}}$ is an isomorphism.

Thus, w.l.o.g., if $\mathcal{M} \models \Phi$, assume $V_\kappa \subseteq M$.

Let L_κ^+ be L_κ with additional constant c and

$$\Phi^* := \Phi \cup \{ c \text{ is an ordinal} \} \cup \{ c \neq c_x ; x \in V_\kappa \}$$

<u>Claim</u> Φ^* is κ-satisfiable

[If $\Phi_0 \subseteq \Phi^*$, $|\Phi_0| < \kappa$, then there are ordinals α s.t. $c \neq c_\alpha \notin \Phi_0$. Interpret c by α in $\mathcal{V}$.]

So by weak compactness of κ, we get a model $\mathfrak{M} \models \Phi^*$.

By ②, $V_\kappa \subseteq M$.

By ①, $\mathfrak{M}$ is wellfounded.

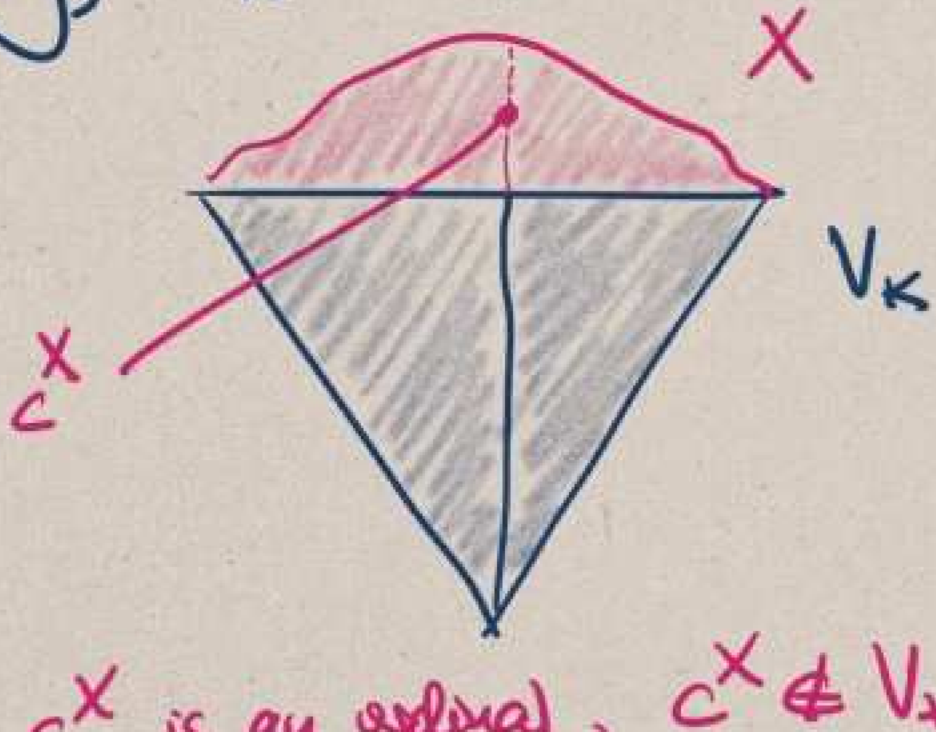
The analogue of 'subset collapse' is:

Theorem 4 (Mostowski's Collapsing Theorem). Let r be a relation on a set a that is well-founded and extensional. Then there exists a transitive set b, and a bijection $f : a \to b$ such that $(\forall x, y \in a)(x \; r \; y \Leftrightarrow f(x) \in f(y))$. Moreover, b and f are unique.

Remark. 'Well-founded' and 'extensional' are trivially necessary.

So by Mostowski, there is transitive X s.t. $(X, \in) \cong (M, E)$.

Clearly, $V_\kappa \subseteq X$.



c^X is an ordinal, $c^X \notin V_\kappa$, $c^X \geq \kappa$.

$\Longrightarrow \kappa \in X$.

[since X is transitive]

Prove that $V_\kappa \preccurlyeq X$:

Suppose φ is first-order, $x_1, \dots, x_n \in V_\kappa$

$$V_\kappa \models \varphi(x_1, \dots, x_n) \iff \gamma \models \varphi(c_{x_1}, \dots, c_{x_n})$$

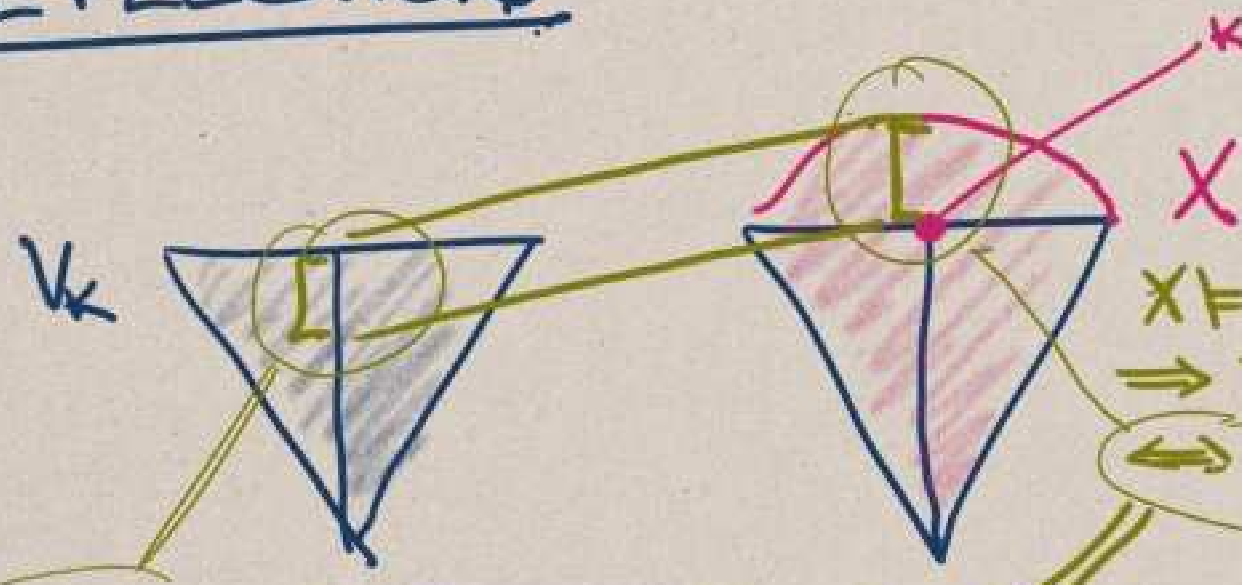
$$\iff \varphi(c_{x_1}, \dots, c_{x_n}) \in \Phi$$

$$\iff (X, \in, \{c_x^X ; x \in V_\kappa\}) \models \varphi(c_{x_1}, \dots, c_{x_n})$$

$$\iff X \models \varphi(x_1, \dots, x_n).$$

q.e.d.

REFLECTION



$X \models \kappa$ is inaccessible

$\Rightarrow X \models IC$

$\iff X \models \exists \alpha (\alpha$ is inaccessible$)$

by elementarity

$V_\kappa \models \exists \alpha (\alpha$ is inaccessible$)$

This process is called

REFLECTION.

Reflection allows for bootstrapping as follows:

Consider

$$\varphi := \exists \nu (\nu > \omega \wedge \nu \text{ is an inaccessible cardinal})$$

Fix $\alpha < \kappa$, then

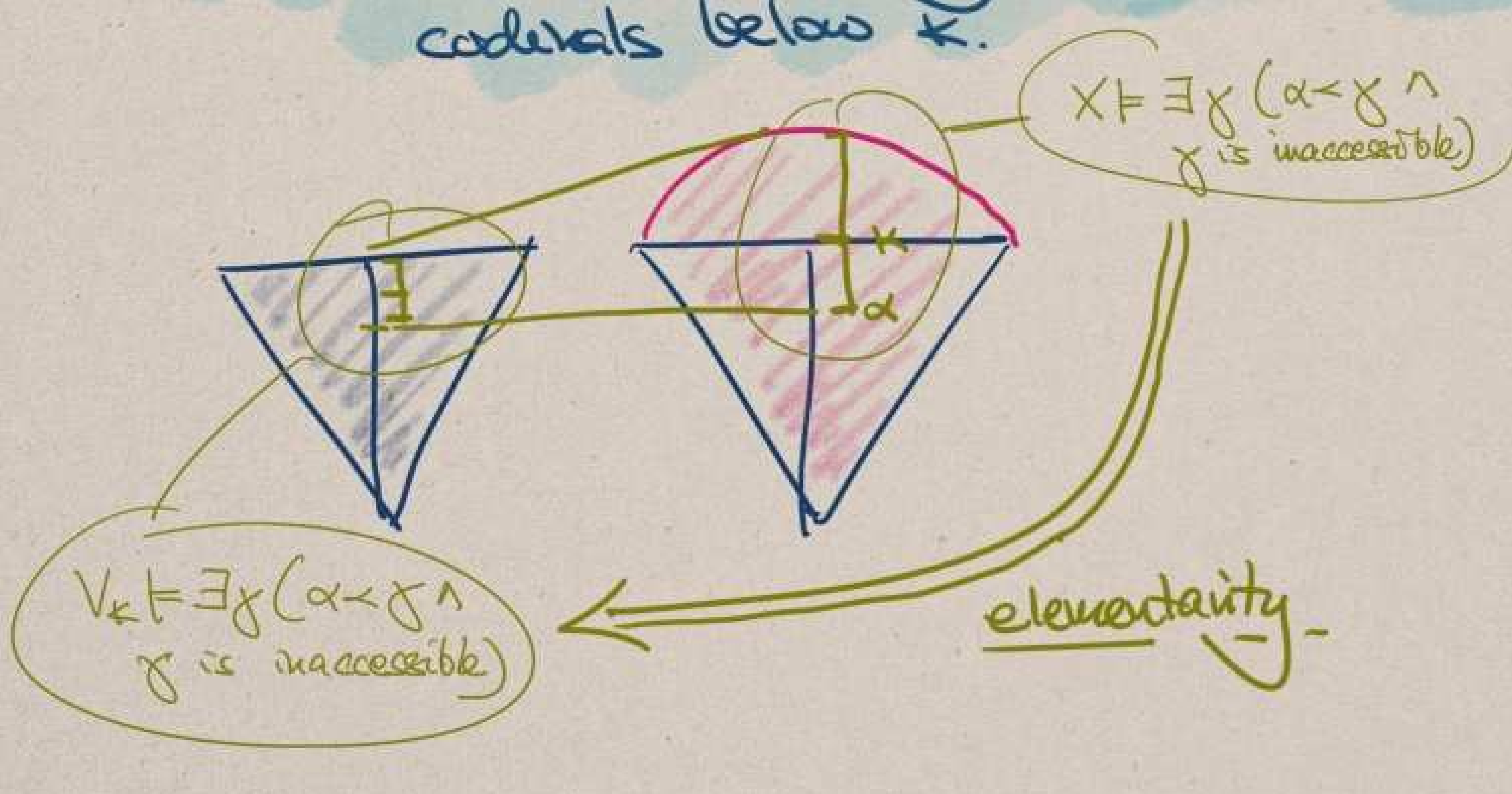
$$X \models \varphi(\alpha),$$

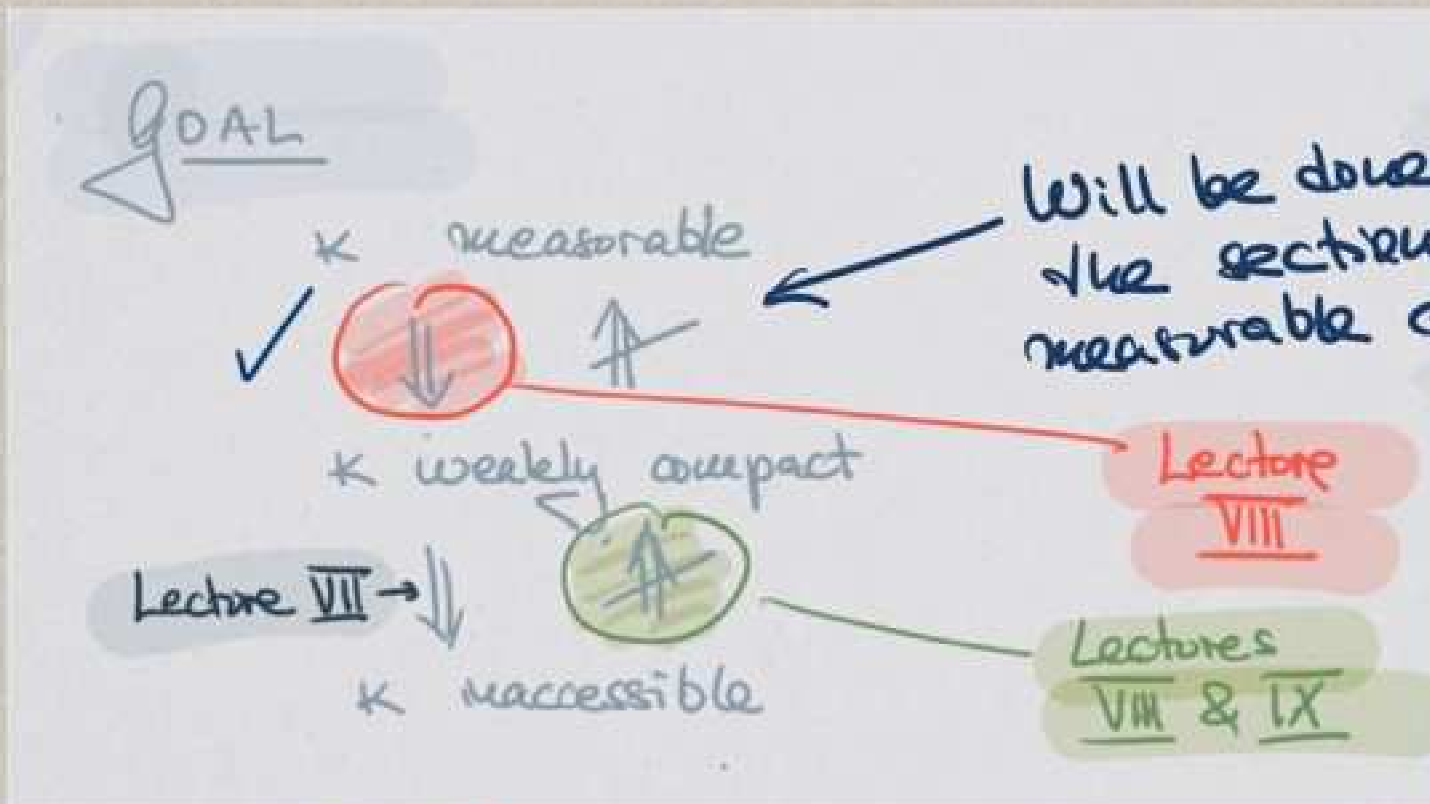
thus by elementarity

$$V_\kappa \models \varphi(\alpha).$$

So, the set of inaccessibles below κ is unbounded.

<u>Corollary</u> If κ is weakly compact, then there are κ many inaccessible cardinals below κ.





Another LCA from combinatorics.


Paul ERDŐS
1913–1996

ERDŐS ARROW NOTATION.

Let κ, λ, μ be cardinals, $m \in \mathbb{N}$. If $X \subseteq \kappa$, write $[X]^m$ for the set of m-element subsets of X.

$$\kappa \longrightarrow (\lambda)^m_\mu$$ "κ arrows $(\lambda)^m_\mu$"

$$:\Longleftrightarrow \forall f : [\kappa]^m \longrightarrow \mu$$ colourings with μ colours

$$\exists H \subseteq \kappa \quad |H| = \lambda$$

homogeneous / monochromatic

s.t. there is $\alpha < \mu$ and for all $x \in [H]^m$, $f(x) = \alpha$.

Observe:

1. $\omega \longrightarrow (\omega)^m_n$ is Ramsey's theorem

2. Note that the Erdős arrow notation is "upwards monotone" on the LHS "downwards monotone" on the RHS $\longrightarrow$ ES #2.

<u>Def.</u> κ is called <u>finite partition cardinal</u> if $\kappa \longrightarrow (\kappa)^2_2$.

We'll show:

κ measurable

Lecture X

$\Longrightarrow$ κ finite partition

PROVED ON PP. 9 & 10 OF LECTURE IX

$\Longrightarrow$ κ inaccessible.

<u>In fact</u>. <u>THEOREM</u> (without proof)

κ is weakly compact

$\Longleftrightarrow$

κ is finite partition.

Proposition κ finite partition $\Longrightarrow$
κ inaccessible

Pf. REGULAR Towards a contradiction,
assume $\kappa = \bigcup_{\alpha<\lambda} A_\alpha$ $\quad \lambda < \kappa$
A_α disjoint
$|A_\alpha| < \kappa$.

Define $f(\gamma,\delta) := \begin{cases} 1 & \text{if } \exists \alpha \text{ s.t. } \gamma,\delta \in A_\alpha \\ 0 & \text{o/w}. \end{cases}$

$\kappa \longrightarrow (\kappa)^2_2$ implies that there is $H \subseteq \kappa$,
$|H| = \kappa$, H is monochromatic.

If it's monochromatic for 1, then there is α s.t. $H \subseteq A_\alpha$. Contradiction to $|H| = \kappa$
$|A_\alpha| < \kappa$.

If it's monochromatic for 0, then
$x \longmapsto \alpha$ where $x \in A_\alpha$
is an inj. func H into λ, but that contradicts $\lambda < \kappa$
$|H| = \kappa$.

So κ is regular.

STRONG LIMIT Let $\lambda < \kappa$. Need to show $2^\lambda < \kappa$.

Towards a contradiction, assume $2^\lambda \geq \kappa$. $(*)$

Consider 2^λ as $\{f ; f : \lambda \longrightarrow \{0,1\}\}$.
This can be ordered lexicographically by

$$f <_{lex} g \;:\Longleftrightarrow\; f(\alpha) = 0 \text{ and } g(\alpha) = 1 \text{ for } \alpha \text{ the least s.t. } f(\alpha) \neq g(\alpha).$$

Combinatorial fact (ES#2):
There are no length κ strictly increasing or decreasing sequences in $(2^\lambda, <_{lex})$.

By assumption $(*)$, let
$\{f_\alpha ; \alpha < \kappa\} \subseteq 2^\lambda$ be a family of functions.

Define $F(\alpha,\beta) := \begin{cases} 1 & \text{if } \alpha<\beta \leftrightarrow f_\alpha <_{lex} f_\beta \\ 0 & \text{o/w} \end{cases}$

By $\kappa \longrightarrow (\kappa)^2_2$, there is a homogeneous set of size κ. Homogeneity for 1 gives a strictly increasing seq. of length κ; homogeneity for 0 a strictly decreasing seq. q.e.d.