IV

FOURTH LECTURE LARGE CARDINALS

1 February 2023

RECAP

$I(\kappa) :\Longleftrightarrow \kappa$ is regular strong limit

Strength of $I(\kappa)$ derives from the fact that $V_\kappa \models ZFC$ [more precisely, V_κ satisfies SOR]

$W(\kappa) :\Longleftrightarrow V_\kappa \models ZFC$

WORLDLY CARDINAL

Clearly (by Zermelo's Thm), $I(\kappa) \Rightarrow W(\kappa)$.

Q: Does the converse hold? NO!

Goal for today: $I(\kappa) \Longrightarrow \{\lambda < \kappa ; W(\lambda)\}$ is unbounded in κ.

This implies:

1. Since κ is regular, there are κ many worldly cardinals below κ.
2. So the smallest inaccessible cannot be the smallest worldly.
3. In particular $W(\kappa) \not\Rightarrow I(\kappa)$.

First check that $W(\kappa)$ is a LARGE CARDINAL AXIOM:

<u>Proposition</u> $W(\kappa) \Longrightarrow \kappa$ is a cardinal.

<u>Proof.</u> Suppose not, then there is $\lambda < \kappa$ s.t. $f : \lambda \longrightarrow \kappa$ is a bijection.

Define
$$R := \{(\alpha, \beta) \in \lambda^2 ; f(\alpha) < f(\beta)\}$$

By construction
$$(\lambda, R) \cong (\kappa, <)$$ via the function f.

$(\lambda, R) \in V_{\lambda+10} \subseteq V_\kappa$.

[Note that $V_\kappa \models ZFC$, in particular $V_\kappa \models$ "there is no largest ordinal", so κ is a limit ordinal.]

Since $V_\kappa \models ZFC$, it satisfies the REPRESENTATION THM FOR WELLORDERS (every w.o. is isom. to a unique ordinal)

(λ, R) is a wellorder; it can only be isom. to κ, so $\kappa \in V_\kappa$.

Contradiction! q.e.d.

Similar proof (Example Sheet #1):

$W(\kappa) \Longrightarrow \kappa$ is a limit cardinal

&

$W(\kappa) \Longrightarrow \kappa$ is an aleph fixed pt

In a moment, we'll see: not quite as large as we would like since they can be singular (actually, most of them are).

BASIC MODEL-THEORETIC CONCEPTS

Let M, N be $\mathcal{L}$-structures.

$M \equiv N$ — M & N are elementarily equivalent

$:\Longleftrightarrow \forall \sigma$ $\mathcal{L}$-sentence

$M \models \sigma \Longleftrightarrow N \models \sigma$

$M \preccurlyeq N$ — M is an elementary substructure of N

$:\Longleftrightarrow M \subseteq N$ & $\forall \varphi$ $\mathcal{L}$-fmla with n free variables & all $\vec{a} \in M^n$

$M \models \varphi(\vec{a}) \Longleftrightarrow N \models \varphi(\vec{a})$

Note that $M \preccurlyeq N \Longrightarrow M \equiv N$

In general, $M \preccurlyeq N$ is much stronger.

$\pi : M \longrightarrow N$ is an elementary embedding

$:\Longleftrightarrow \forall \varphi$ $\mathcal{L}$-fmla w/ n free variables & all $\vec{a} \in M^n$

$M \models \varphi(\vec{a}) \Longleftrightarrow N \models \varphi(\pi(\vec{a}))$.

In particular, if φ identifies a unique object, e.g., the empty set, ω, the smallest inaccessible, then this object will be THE SAME in M & N.

BASIC MODEL-THEORETIC RESULTS:

(1) TVT Tarski-Vaught Test

Marker, Model Theory, p. 45

Proposition 2.3.5 (Tarski–Vaught Test) *Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then, $\mathcal{M}$ is an elementary substructure if and only if, for any formula $\phi(v, \overline{w})$ and $\overline{a} \in M$, if there is $b \in N$ such that $\mathcal{N} \models \phi(b, \overline{a})$, then there is $c \in M$ such that $\mathcal{N} \models \phi(c, \overline{a})$.*

<u>TEST</u>: If $\mathcal{N} \models \exists x \varphi(x, \vec{a})$ then a witness of this lies already in M.

[Proof sketch: Induction on complexity of φ. Only one case in the induction makes trouble: this is taken care of by the condition of the test.]

ES#1

<u>Remark</u>. If $M \preccurlyeq V_\kappa$ with $W(\kappa)$, then $M \models ZFC$.

If $V_\lambda \preccurlyeq V_\kappa$ and $W(\kappa)$, then $W(\lambda)$.

So, we'll show $\{\lambda < \kappa ; V_\lambda \preccurlyeq V_\kappa\}$ is unbounded.

② If $\left.\begin{array}{l} M_0 \preccurlyeq N \\ M_1 \preccurlyeq N \\ M_0 \subseteq M_1 \end{array}\right\} M_0 \preccurlyeq M_1$

[Just the definitions.]

So, if $V_\lambda \preccurlyeq V_\kappa$ & $V_{\lambda'} \preccurlyeq V_\kappa$, then $V_\lambda \preccurlyeq V_{\lambda'}$.
$\lambda < \lambda'$

③ TARSKI'S CHAIN LEMMA

Let $(L, <)$ be a total order and M_ℓ be an $\mathcal{L}$-structure for all $\ell \in L$.
Then $(M_\ell ; \ell \in L)$ is called <u>elementary chain</u> if for all $\ell < \ell'$, $M_\ell \preccurlyeq M_{\ell'}$.

If $(M_\ell ; \ell \in L)$ is an elem. chain and

$$M := \bigcup_{\ell \in L} M_\ell$$

then for all $\ell \in L$, $M_\ell \preccurlyeq M$.

[Proof on ES#1, simple adaptation of TVT.]

Theorem $I(\kappa) \Rightarrow \{\lambda < \kappa ; V_\lambda \preccurlyeq V_\kappa\}$ is unbounded.

Proof. Fix any $\gamma < \kappa$; find $\lambda > \gamma$ s.t. $V_\lambda \preccurlyeq V_\kappa$.

Idea: TVT, so make sure that all existential fmlas have witnesses in V_λ.

Consider any V_α. In order to do the TVT, I need to check every pair $(\varphi, \vec{a})$ where φ is a formula and $\vec{a} \in V_\alpha^{<\omega}$ and $V_\kappa \models \exists x\, \varphi(x, \vec{a})$ and find a witness $w(\varphi, \vec{a})$ s.t.

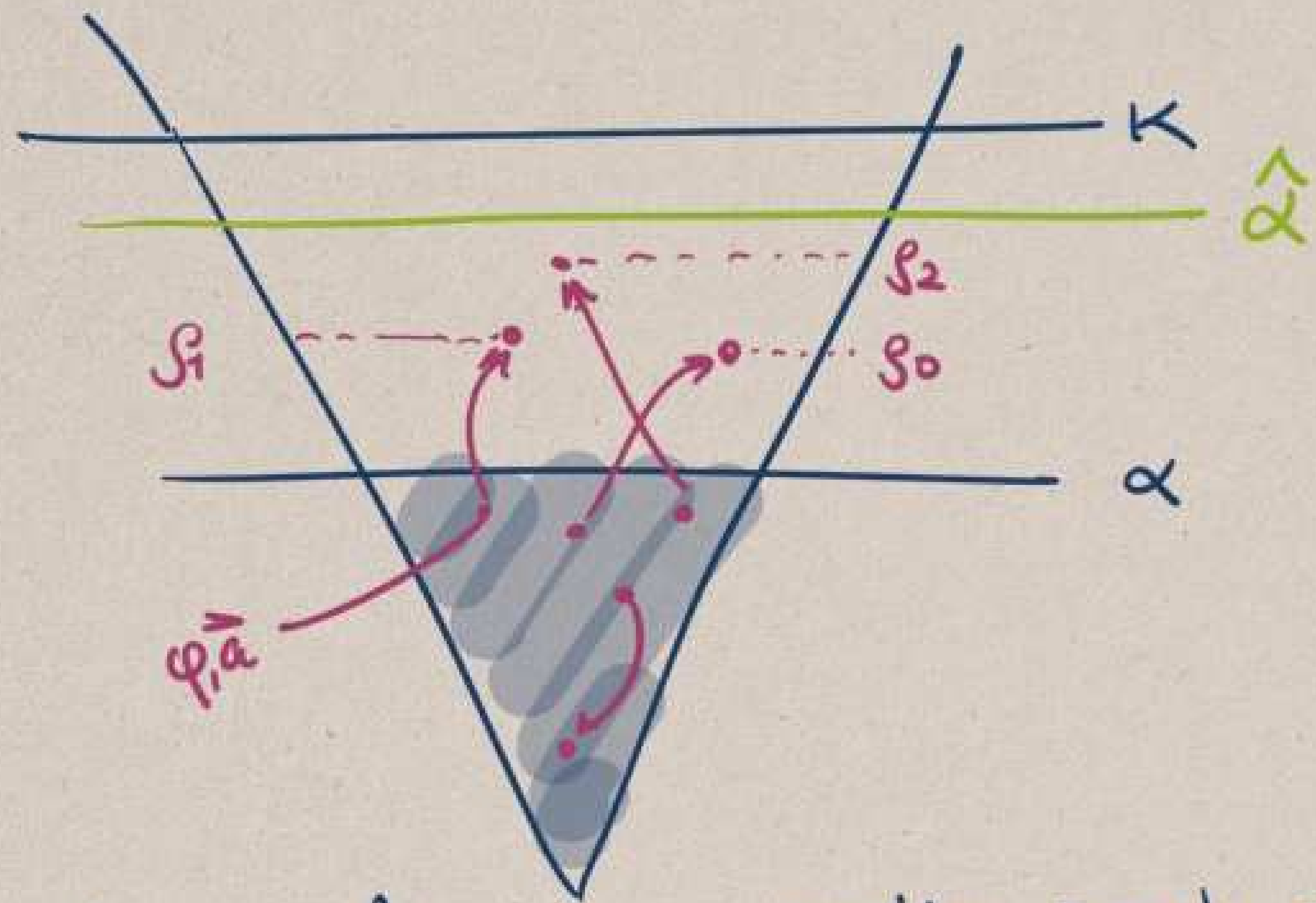
$$V_\kappa \models \varphi(w(\varphi, \vec{a}), \vec{a}).$$

In general, there is no reason to assume $w(\varphi, \vec{a}) \in V_\alpha$.

How many things do we need to check?

$$|\underbrace{\mathrm{Fml}}_{\text{countable}} \times \underbrace{V_\alpha^{<\omega}}_{\text{size } |V_\alpha| < \kappa}| < \kappa$$

since κ was inaccessible.



$R := \{\rho \,;\, \rho$ is the rank of some $w(\varphi, \vec{a})$ for $\vec{a} \in V_\alpha^{<\omega}\}$

$|R| < \kappa \quad R \subseteq \kappa$

So, by regularity, R is bounded.

Let $\quad \alpha_0 := \gamma + 1$

$\alpha_{i+1} := \max(\alpha_i + 1, \hat{\alpha}_i)$

$\lambda := \sup\{\alpha_i \,;\, i \in \omega\}$

This is to make sure that the seq. is strictly increasing and λ is a limit ordinal.

Claim $V_\lambda \preccurlyeq V_\kappa$

[λ satisfies the TVT:
if $\vec{a} \in V_\lambda^{<\omega}$, there is some $i \in \mathbb{N}$
s.t. $\vec{a} \in (V_{\alpha_i})^{<\omega}$, so

$$w(\varphi, \vec{a}) \in V_{\hat{\alpha}_i} \subseteq V_{\alpha_{i+1}} \subseteq V_\lambda$$

for any fmla φ.]

q.e.d.

Remark All worldly cardinals constructed in this proof have cofinality $\aleph_0$.

Q. Can we get other cofinalities?

(Corollary?)
<u>Theorem</u> If $I(\kappa)$ and $\mu < \kappa$ is regular, then there is a worldly $\lambda < \kappa$ s.t. $\mathrm{cf}(\lambda) = \mu$.

<u>Proof.</u> $W = \{\lambda < \kappa ; V_\lambda \preccurlyeq V_\kappa\}$ is unbounded, so it has size κ, so let $\alpha \longmapsto \lambda_\alpha$ be its increasing enumeration.

MODEL-THEORETIC BASICS

$$\alpha < \beta \quad \begin{matrix} V_{\lambda_\alpha} \preccurlyeq V_\kappa \\ V_{\lambda_\beta} \preccurlyeq V_\kappa \end{matrix} \overset{2}{\Longrightarrow} V_{\lambda_\alpha} \preccurlyeq V_{\lambda_\beta}.$$

So W and all of its subsets form elementary chains.

Consider $\{V_{\lambda_\alpha} ; \alpha < \mu\}$. This is an elem. chain, so $\bigcup_{\alpha<\mu} V_{\lambda_\alpha} = V_\lambda$ for $\lambda = \sup_{\alpha<\mu} \lambda_\alpha$ has the property $V_{\lambda_\alpha} \preccurlyeq V_\lambda$.

Thus $V_\lambda \models \mathrm{ZFC} \Longrightarrow W(\lambda)$.

Regularity of μ implies $\mathrm{cf}(\lambda) = \mu$. q.e.d.

REMARK

While we can show that $W := \{\lambda < \kappa ; W(\lambda)\}$ contains elements of arbitrarily large cofinality, the one thing we cannot show is that it contains a REGULAR element:

Suppose GCH, i.e., every limit cardinal is a strong limit, so every regular limit cardinal is inaccessible.

By ES#1, worldly cardinals are limit cardinals, so if one of them is regular, it is (assuming GCH) inaccessible.

Thus in a model of ZFC + GCH + IC, the smallest inaccessible cardinal has no regular worldlies below it.