III

LARGE CARDINALS
THIRD LECTURE

30 January 2023

RECAP

$I(\kappa) :\Longleftrightarrow \kappa$ is inaccessible

$IC :\Longleftrightarrow \exists \kappa\, I(\kappa)$

1. $I(\kappa) \Longrightarrow \kappa = \aleph_\kappa$
2. $I(\kappa) \Longrightarrow V_\kappa \models ZFC$
3. $I(\kappa) \Longrightarrow \mathrm{Cons}(ZFC + \neg IC)$

   [if κ is least inaccessible, then $V_\kappa \models ZFC + \neg IC$.]

Together:
1. $I(\kappa)$ implies κ is large
2. IC is not provable
3. The largeness of κ features in the proof of 2.

} IC is a LARGE CARDINAL AXIOM

The consistency strength hierarchy.

Fix a base theory $\mathcal{B}$

In our context, we can always assume $\mathcal{B} = \mathrm{ZFC}$.

[Proof-theorists would use some weak arithmetic.]

We consider theories T s.t.

$\mathcal{B} \subseteq T$, T is computable enumerate and deductively closed [if $T \vdash \varphi \Rightarrow \varphi \in T$].

On theories like this, we define

$$T \leq_{\mathrm{Cons}} S \;:\Longleftrightarrow\; \mathcal{B} \vdash \mathrm{Cons}(S) \Rightarrow \mathrm{Cons}(T)$$

$$T \equiv_{\mathrm{Cons}} S \;:\Longleftrightarrow\; T \leq_{\mathrm{Cons}} S \;\&\; S \leq_{\mathrm{Cons}} T$$

$$T <_{\mathrm{Cons}} S \;:\Longleftrightarrow\; T \leq_{\mathrm{Cons}} S \;\&\; S \not\leq_{\mathrm{Cons}} T$$

EQUICONSISTENCY

If T is c.e., so is $T+\varphi$.

<u>Notation</u> If T is a theory & φ is a sentence.

$T+\varphi$ is the deductive closure of $T \cup \{\varphi\}$.

Properties of $\leq_{Cons}$.

1. By deductive closure, there is exactly one inconsistent theory, write

$$\bot$$

for this.

This theory is the maximal element in $\leq_{Cons}$ and we get that

$$T <_{Cons} \bot \iff T \text{ is consistent.}$$

2.

David Hilbert
1862–1943

Königsberg
Radio Address
1930

The HILBERTIAN DREAM ("Finitism"):

There is a base theory $\mathcal{B}$ s.t. φ is true iff $\mathcal{B} \vdash \varphi$.

THIS IS NOT TRUE

The Hilbertian dream was shattered by Gödel's Incompleteness Thm.

If this existed, then of course all consistent theories are equiconsistent.

Thus: The interest in $\leq_{Cons}$ derives from the INCOMPLETENESS PHENOMENON which shows that $\leq_{Cons}$ is nontrivial.

3. $<_{\text{Cons}}$ is stronger than just "proves more theorems".

<u>Example</u> CH.

Proofs by Gödel & Cohen:

1938 $\text{Cons}(ZFC) \Rightarrow \text{Cons}(ZFC + CH)$

1962 $\text{Cons}(ZFC) \Rightarrow \text{Cons}(ZFC + \neg CH)$

Thus $ZFC \equiv_{\text{Cons}} ZFC + CH \equiv_{\text{Cons}} ZFC + \neg CH$

But of course $ZFC + CH$ proves more theorems than ZFC.

4. Gödel's 2nd Incompleteness Theorem implies

if $T \nvdash \bot$, then $T <_{\text{Cons}} T + \text{Cons}(T)$.

5. <u>Careful</u> There are consistent theories T s.t. $T + \text{Cons}(T) = \bot$.

<u>Example</u> G2 implies $ZFC \nvdash \text{Cons}(ZFC)$

[Assuming that ZFC is consistent] or in other words $T := ZFC + \neg\text{Cons}(ZFC)$ is consistent.

Since $T \supseteq ZFC$, $\text{Cons}(T) \Rightarrow \text{Cons}(ZFC)$,

so $T + \text{Cons}(T) \vdash \text{Cons}(ZFC)$

Since $T + \text{Cons}(T) \supseteq T$, $T + \text{Cons}(T) \vdash \neg\text{Cons}(ZFC)$.

So $T + \text{Cons}(T) = \bot$.

6. There is a stronger notion of consistency, called ω-consistency
[essentially adding an infinitary proof rule that corresponds to the naive idea of semantics of "$\forall n \in \mathbb{N}$" and demanding that no contradiction can be derived even with this additional rule].

If T is ω-consistent, then $T + \mathrm{Cons}(T)$ is consistent.

7. Luckily, our proof of $ZFC + IC \vdash \mathrm{Cons}(ZFC)$ gives much more:

$$ZFC_0 := ZFC$$
$$ZFC_{i+1} := ZFC_i + \mathrm{Cons}(ZFC_i)$$

[We could even add
$$ZFC_\omega := \text{"for all } i, ZFC_i\text{"}$$
and continue into the transfinite.]

Want that
$$ZFC_0 <_{\mathrm{Cons}} ZFC_1 <_{\mathrm{Cons}} ZFC_2 <_{\mathrm{Cons}} \cdots$$

We proved $ZFC + IC \vdash \mathrm{Cons}(ZFC)$ by a very concrete model, viz. V_κ, of ZFC.

Since $\omega \subseteq V_\kappa$ and therefore

$\exists n \in \mathbb{N}$ and $\forall n \in \mathbb{N}$

are interpreted the same way in V_κ & V, we get that all formulas whose quantifiers are restricted to $\mathbb{N}$ will get the same truth value in V_κ and V. $V \models \varphi \iff V \models$ "$V_\kappa \models \varphi$"

(ARITHMETICAL FORMULAS)

In other words, arithmetical formulas are absolute between V_κ & V.

<u>IMPORTANT</u> For any T, $\mathrm{Cons}(T)$ is an arithmetical formulas.

<u>BOOTSTRAP</u>: Suppose $V \models IC$ 1

$\Longrightarrow V \models$ "$V_\kappa \models ZFC$" 2

$\Longrightarrow V \models \mathrm{Cons}(ZFC)$ 3

$\xRightarrow{\text{Absoluteness}} V \models$ "$V_\kappa \models \mathrm{Cons}(ZFC)$" 4

$\xRightarrow{2+4} V \models$ "$V_\kappa \models ZFC_1$" 5

$\Longrightarrow V \models \mathrm{Cons}(ZFC_1)$ 6

$\xRightarrow{\text{Absoluteness}} V \models$ "$V_\kappa \models \mathrm{Cons}(ZFC_1)$" 7

$\xRightarrow{5+7} V \models$ "$V_\kappa \models ZFC_2$". 8

AND SO ON.

To summarize:

ZFC+IC proves

$$\mathrm{ZFC}_0 <_{\mathrm{Cons}} \mathrm{ZFC}_1 <_{\mathrm{Cons}} \cdots <_{\mathrm{Cons}} \mathrm{ZFC}_i <_{\mathrm{Cons}} \cdots < \mathrm{ZFC}+\mathrm{IC}$$

Natural question

Is this about inaccessibility or rather about $V_\kappa \models \mathrm{ZFC}$?

Definition An ordinal α is called WORLDLY if $V_\alpha \models \mathrm{ZFC}$.

Are "inaccessible" and "worldly" the same thing.

Answer will be (Lectures IV & V):

NO!

In particular, we'll show that if $I(\kappa) \Rightarrow$ there are many worldly cardinals below κ.

The difference between INACCESSIBLE and WORLDLY cardinals comes down to the difference between

$V_\kappa \models$ Replacement

and

V_κ satisfies SOR

mentioned in Lecture II.