LARGE CARDINALS

II SECOND LECTURE
25 January 2023

RECAP Lecture I, page 3

A LCA is of the form $\exists\kappa\,\Phi(\kappa)$ [we write ΦC for this] where Φ is a property of cardinals that implies that κ is "very large" and this largeness results in ΦC not being provable in ZFC.

All three components are needed: very large; not provable; largeness being the reason for lack of provability.

Non-examples

Φ : "is a counterexample to GCH"
(1) existence not provable in ZFC
(2) not necessarily large

Ψ : "is an aleph fixed point"
(1) very large
(2) existence provable in ZFC

This is an example s.t. ZFC $\nvdash \Xi C$ and $\Xi(\kappa) \Rightarrow \kappa$ is large, but the largeness is not the reason for ZFC $\nvdash \Xi C$.

Ξ : "is an aleph fixed pt larger than a counterex to GCH".

Further RECAP

κ REGULAR : if $C \subseteq \kappa$ cofinal, then $|C| = \kappa$

κ regular $\Longrightarrow$ if $\lambda < \kappa$, $|X_\alpha| < \kappa$ for all $\alpha < \lambda$, then $|\bigcup X_\alpha| < \kappa$. (*)

κ STRONG LIMIT : if $\lambda < \kappa$, then $2^\lambda < \kappa$.

κ INACCESSIBLE : κ regular strong limit

$I(\kappa) :\Longleftrightarrow \kappa$ is inaccessible

Prop. $I(\kappa) \Longrightarrow \kappa$ is an aleph fixed point

[Remark: this only used regular limit.!!]

NOW: IC is not provable in ZFC

[Remark: We mean: $\mathrm{Cons(ZFC)} \Longrightarrow \mathrm{ZFC} \nvdash \mathrm{IC}$.]

Gödel 2nd Incompleteness Theorem (G2)

If T is an "appropriate" foundational theory, then

if T is consistent, $T \nvdash \mathrm{Con}(T)$.

Gödel's Completeness Theorem implies

(*) $T \vdash \mathrm{Cons}(S) \iff T \vdash$ "$\exists M\ M \models S$".

Therefore, in order to show

$$\mathrm{ZFC} \nvdash \varphi \quad [\text{under ass. } \mathrm{Cons}(\mathrm{ZFC})]$$

it is enough to show

$$\mathrm{ZFC} + \varphi \vdash \exists M\ (M \models \mathrm{ZFC}).$$

[Why? $\mathrm{ZFC} + \varphi \vdash \exists M\ (M \models \mathrm{ZFC})$

$\overset{(*)}{\iff}$ $\mathrm{ZFC} + \varphi \vdash \mathrm{Cons}(\mathrm{ZFC})$

$\underset{\text{Ass. } \mathrm{ZFC} \vdash \varphi}{\iff}$ $\mathrm{ZFC} \vdash \mathrm{Cons}(\mathrm{ZFC})$

Contradiction to G2.]

Reminder

VON NEUMANN hierarchy

The V_α are transitive: $x \in V_\alpha \Rightarrow x \subseteq V_\alpha$.

$$V_0 := \emptyset$$

$$V_{\alpha+1} := \mathcal{P}(V_\alpha)$$

$$V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$$

John von Neumann
1903 – 1957

The <u>von Neumann ranks</u> are themselves models of some set theory.

LOGIC & SET THEORY, EXAMPLE SHEET 4

7. A set x is called *hereditarily finite* if each member of $TC(\{x\})$ is finite. Prove that the class HF of hereditarily finite sets coincides with V_ω. Which of the axioms of ZF are satisfied in the structure HF (i.e. the set HF, with the relation $\in |HF$)?

8. Which of the axioms of ZF are satisfied in the structure $V_{\omega+\omega}$? (2022)

If λ is a limit ordinal, $V_\lambda \models$ all of ZFC minus possibly Infinity & Replacement

If $\lambda > \omega$, $V_\lambda \models$ Infinity

If $\lambda = \omega$, $V_\lambda \models$ Replacement.

[This proof is very similar to what we'll see as ZERMELO's thm later.]

Goal. Show that $I(\kappa) \Longrightarrow V_\kappa \models$ Replacement.

So: $I(\kappa) \Longrightarrow V_\kappa \models ZFC$.

Thus: $ZFC \nvdash IC$.

Lemma 1 $I(\kappa), \lambda < \kappa \Longrightarrow |V_\lambda| < \kappa$.

$V_0 := \emptyset$

$V_{\alpha+1} := \wp(V_\alpha)$

$V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$

Proof. Induction.

$|V_0| = 0 < \kappa$. ✓

Suppose $|V_\alpha| < \kappa$, then

$\alpha \mapsto \alpha+1$ $\quad |V_{\alpha+1}| = |\wp(V_\alpha)| = 2^{|V_\alpha|}$

$< \kappa$ by κ strong limit.

γ limit $< \lambda$: $\quad |V_\gamma| = \left|\bigcup_{\alpha<\gamma} V_\alpha\right| < \kappa$

by regularity of κ.

[Uses the property marked as (*) on page 2.]

q.e.d.

Lemma 2 $I(\kappa),\ x \in V_\kappa \Longrightarrow |x| < \kappa.$

Pf. Since κ is a cardinal, therefore a limit ordinal, we have

$$V_\kappa = \bigcup_{\alpha<\kappa} V_\alpha.$$

So, there is $\alpha < \kappa$ s.t. $x \in V_\alpha$.

By transitivity of V_α, $x \subseteq V_\alpha$.

$$|x| \leq |V_\alpha| < \kappa.$$

Lemma 1. q.e.d.

REPLACEMENT

Repl. is a complex axiom [scheme] saying:

If a formula behaves like a function and x is a set then what can be describes a the image of x under the formula is a set.

SOR [Second order replacement]

V_κ is said to satisfy SOR if

$$\forall F : V_\kappa \longrightarrow V_\kappa \ \forall x \in V_\kappa$$

$$F[x] = \{F(y) ; y \in x\} \in V_\kappa .$$

Replacement is the "definable" version of SOR and thus

V_κ satisfies SOR $\longrightarrow V_\kappa \models$ Repl.

Strategy: Prove that $I(\kappa) \Longrightarrow$ V_κ satisfies SOR.

[We will discuss the difference between SOR and $V_\kappa \models$ Repl in the following lectures in detail.]

Ernst ZERMELO
1871–1953

Theorem (ZERMELO's Theorem)

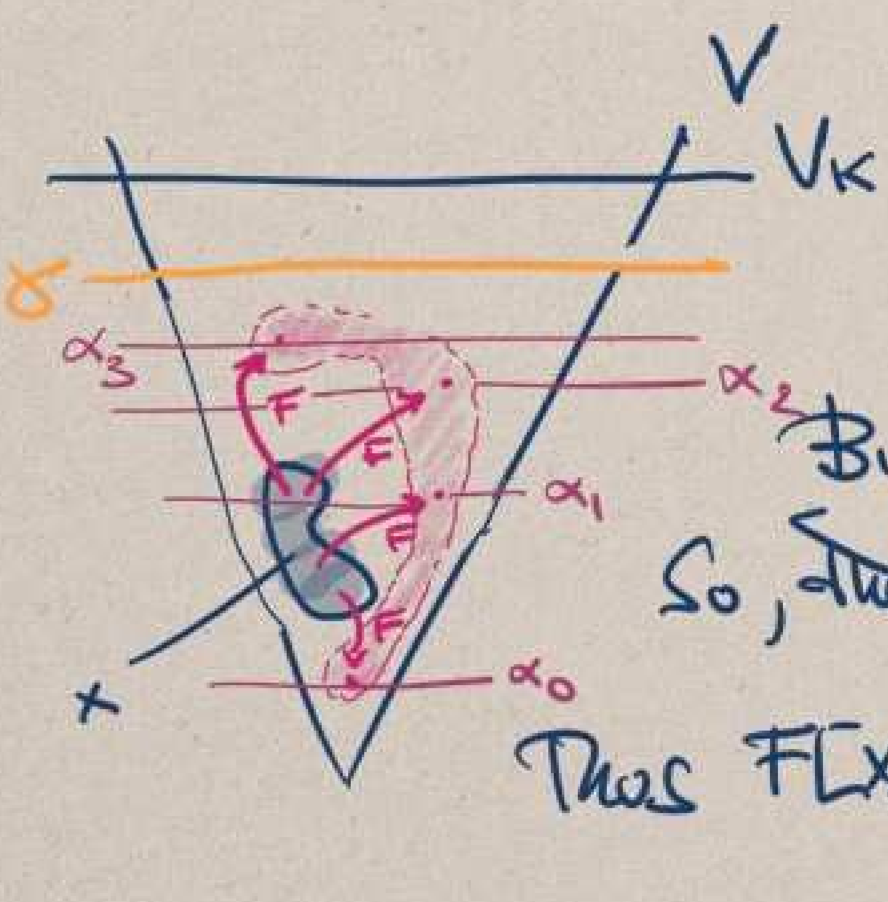
$I(\kappa) \Longrightarrow V_\kappa$ satisfies SOR

Proof. Suppose $F : V_\kappa \longrightarrow V_\kappa$, $x \in V_\kappa$.

Clearly $|F[x]| \leq |x| < \kappa$ by L2.

$\therefore \quad F[x] \subseteq V_\kappa$.

$A := \{ \alpha < \kappa \,;\, \exists z \in F[x]$ s.t. $\mathrm{rank}(z) = \alpha \}$



$A \subseteq \kappa$

$|A| \leq |F[x]| < \kappa$

By regularity, A is not cofinal.

So, there is some $\gamma_{<\kappa}$ s.t. $A \subseteq \gamma$.

Thus $F[x] \subseteq V_\gamma$, so $F[x] \in \mathcal{P}(V_\gamma) = V_{\gamma+1} \subseteq V_\kappa$ q.e.d.

Alternative proof of ZFC $\nvdash$ IC without using G2.

Suppose $\lambda < \kappa$.

λ is regular $\iff V_\kappa \models \lambda$ is regular

[because all of the witnesses of singularity of $\lambda < \kappa$ live in V_κ]

λ is strong limit $\iff V_\kappa \models \lambda$ is strong limit

[same reason]

Thus: <u>L3</u> $\quad I(\lambda) \iff V_\kappa \models I(\lambda)$.

"Inaccessibility is absolute between V_κ & the universe".

If $M \subseteq N$ are two models and φ is a formula, we say φ is <u>absolute</u> between M and N if

$\forall x \in M \ \ M \models \varphi(x) \iff N \models \varphi(x)$

Proof of $ZFC \nvdash IC$:

Need to assume ZFC is consistent: let $V \models ZFC$.

Towards contradiction, assume $ZFC \vdash IC$!.

Thus: $V \models ZFC + IC$.

Let κ be the least inaccessible cardinal.

By Zermelo's Thm, $V_\kappa \models ZFC$.

Thus, by ass. $V_\kappa \models ZFC + IC$.

Thus there is $\lambda < \kappa$ s.t.

$$V_\kappa \models I(\lambda).$$

So by absoluteness, $I(\lambda)$, in contradiction to minimality of κ.

q.e.d.

This proof gives us a concrete model of ZFC + $\neg$IC from a model of ZFC+IC [viz. V_κ for κ the least inaccessible].