Large Cardinals

Lent Term 2023
Part III

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Large Cardinals

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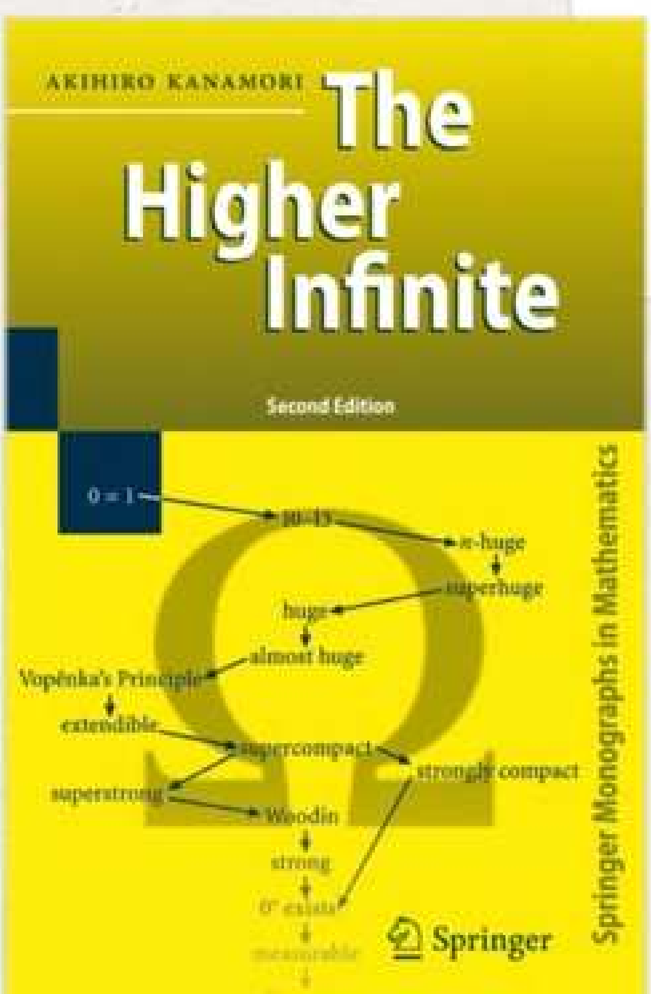
These notes are produced entirely from the course I took and my subsequent thoughts. They are not necessarily an accurate representation of the material presented, and may in places have been substantially edited. Please send any corrections to pdtwm2@cam.ac.uk; any mistakes are almost certainly my own.

Lecture Notes taken by P.M. for Lent 2022 course; available on moodle.



Large Cardinals ?

Small cardinals:
$\mathbb{N}, \aleph_0, \aleph_1, \aleph_2, \ldots$

Better title: "Large Cardinal Axioms"

INCOMPLETENESS PHENOMENON:

For every foundational T, there are φ s.t. $T \nvdash \varphi$
$T \nvdash \neg\varphi$.

This is true for ZFC which happens to be a rather haphazard collection of "obvious" truths about the concept of set.

LCA

Large Cardinal axioms are the most promising candidates for finding more "obvious" truths!

NOT: We will not consider arguments for or against LCA, but study these..

Arguments for or against would be covered in a philosophy of maths course.

Basic principle

ω is very big compared to every $\alpha < \omega$

Wouldn't it be odd if it was the only example of this?

Roughly speaking, a LCA takes a

LARGENESS PROPERTY

of ω and postulates that there is another (uncountable) cardinal with that property.

Slightly more precisely:

A LCA is of the form $\exists \kappa\, \Phi(\kappa)$ [we write ΦC for this] where Φ is a property of cardinals that implies that κ is "very large" and this largeness results in ΦC not being provable in ZFC.

All three components are needed: very large; not provable; largeness being the reason for lack of provability.

REMINDER

CH $\qquad 2^{\aleph_0} = \aleph_1$

CONTINUUM HYPOTHESIS

Theorem Gödel 1938 $\quad ZFC \nvdash \neg CH$ (†)
Cohen 1962 $\quad ZFC \nvdash CH$ (*)

Kurt Gödel
1906–1978

Paul Cohen
1934–2007

The content of this slide with proofs is an entire 24-hour Part III course.

GCH $\qquad \forall \alpha \quad 2^{\aleph_\alpha} = \aleph_{\alpha+1}$

GENERALISED CONTINUUM HYPOTHESIS

Theorem (a) Gödel 1938 $\quad ZFC \nvdash \neg GCH$
(b) Cohen 1962 $\quad ZFC \nvdash GCH$

(b) follows directly from (*); (a) is implicit in the method Gödel used to prove (†)

Two non-examples

(1) $\Phi(\kappa) :\iff$ there is α s.t.

$$\kappa = 2^{\aleph_\alpha} > \aleph_{\alpha+1}.$$

$$\Phi C \iff \exists \kappa\, \Phi(\kappa)$$
$$\iff \neg GCH.$$

Thus, $ZFC \nvdash \Phi C$.

However, it is not a "large cardinal axiom"

The smallest Φ-cardinal could be as small as $\aleph_2$.

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(2) Definition A cardinal κ is called an ALEPH FIXED POINT if

$$\kappa = \aleph_\kappa.$$

Do these things exist at all?

If they exist, they must be huge!

$\Phi(\kappa) :=$ κ is an aleph fixed point

Consider $\Phi C \iff \exists \kappa\ \Phi(\kappa)$.

REMINDER A class function $F : \mathrm{Ord} \to \mathrm{Ord}$ is called NORMAL if

(1) $\forall \alpha, \beta\ (\alpha < \beta \longrightarrow F(\alpha) < F(\beta))$

(2) $\forall \lambda\ (\lambda \text{ limit} \longrightarrow F(\lambda) = \bigcup_{\alpha < \lambda} F(\alpha))$

CONTINUITY

An ordinal α is called a fixed point of F if

$$F(\alpha) = \alpha.$$

Theorem Any normal ordinal operation has arbitrarily large fixed pts.

Pf. Fix γ and find $\alpha > \gamma$ s.t. $F(\alpha) = \alpha$.

Apply this to the aleph function & $\gamma = \omega$:

$\alpha_0 = \omega$ $\quad \alpha_2 = \aleph_{\aleph_0}$

$\alpha_1 = \aleph_\omega$ $\quad \alpha_3 = \aleph_{\aleph_{\aleph_\omega}}$ ...

$$\alpha_0 := \gamma$$
$$\alpha_{i+1} := F(\alpha_i)$$
$$\alpha := \bigcup_{i \in \mathbb{N}} \alpha_i .$$

$\alpha = \bigcup_{k \in \mathbb{N}} \aleph_{\aleph_{\ldots \aleph_\omega}}$ the SMALLEST ALEPH FIXED POINT

$cf(\alpha) = \aleph_0$

Claim $F(\alpha) = \alpha$. Clearly, α is a limit ordinal

$$F(\alpha) = \bigcup_{\beta < \alpha} F(\beta)$$

by continuity

$$= \bigcup_{i \in \mathbb{N}} F(\alpha_i)$$
$$= \bigcup_{i \in \mathbb{N}} \alpha_{i+1} = \alpha .$$

q.e.d.

Corollary ZFC proves that aleph fixed pts exist: $ZFC \vdash \Phi C$.

So: ΦC is not a LCA.

Reminder

Closure properties
Largeness properties } of ω

κ^+ is the smallest ordinal not injecting into κ [the HARTOGS ALEPH OF κ]

$n^+ = n+1$

$\aleph_\alpha^+ = \aleph_{\alpha+1}$

2^κ is the unique cardinal in bijection with $\wp(\kappa) := \{A ; A \subseteq \kappa\}$.

In general, $\kappa^+ \leq 2^\kappa$.

Note: ω is closed under $\alpha \longmapsto \alpha^+$
[$\iff$ limit cardinal]

ω is closed under $\kappa \longmapsto 2^\kappa$
[call this: strong limit cardinals]

If κ is a cardinal, $C \subseteq \kappa$, say that C is COFINAL / UNBOUNDED if

$\forall \alpha \in \kappa \, \exists \gamma \in C \, (\gamma > \alpha)$.

$cf(\kappa) := \min\{|C| ; C \text{ is cofinal in } \kappa\}$

$cf(\kappa) \leq \kappa$

Example

$$\aleph_\omega = \bigcup_{n\in\mathbb{N}} \aleph_n$$

so $C = \{\aleph_n;\, n\in\mathbb{N}\} \subseteq \aleph_\omega$ s.t.

$\bigcup C = \aleph_\omega$. [i.e., cofinal]

$cf(\aleph_\omega) = \aleph_0$

Def. κ is called REGULAR if

$$cf(\kappa) = \kappa$$

SINGULAR

$$cf(\kappa) < \kappa.$$

Clearly, ω is regular.

Definition κ is called (STRONGLY) INACCESSIBLE if it is uncountable, regular and a strong limit.

Write: $I(\kappa) \rightsquigarrow IC$ "there are inaccessible cardinals"

Proposition If $I(\kappa)$, then κ is an aleph fixed pt.

Proof. Suppose not. $\kappa < \aleph_\kappa$.

Note that this proof does not really need "strong limit"; regular limit is enough.

Let λ be s.t. $\kappa = \aleph_\lambda$.
$\lambda < \kappa$

Since κ is limit cardinal, λ must be a limit ordinal, so.

$$\aleph_\lambda = \bigcup \{\aleph_\alpha ; \alpha < \lambda\}.$$

Therefore

$$cf(\kappa) = cf(\aleph_\lambda) \leq |\lambda| < \kappa.$$

Thus κ is singular. q.e.d

In Lecture #2, we'll show that

$$ZFC \nvdash IC.$$

Whenever we say "we prove $ZFC \nvdash \varphi$" we of course mean "if ZFC is consistent, then $ZFC \nvdash \varphi$".